Week 7: Introduction to Quantum Fourier Transform and Midterm Review

COMS 4281 (Fall 2025)

Admin

1. Midterm on Thursday, October 16. You will be assigned an exam room (either Havemeyer 209 or Hamilton 703).
2. If you miss the midterm, you can make it up by scheduling a 15-minute oral exam with me.

Quantum Fourier Transform

Quantum Fourier Transform

Quantum algorithm that implements the Discrete Fourier Transform (DFT) on exponentially large dimension.

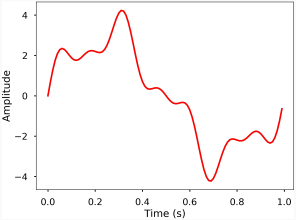
It is the heart of many powerful quantum algorithms such as

- Shor's factoring algorithm
- Phase estimation algorithm
- Algorithms for solving hidden subgroup problem

Discrete Fourier Transform

A method to uncover **hidden, periodic structure** in vectors. Used everywhere in engineering, science, and mathematics.

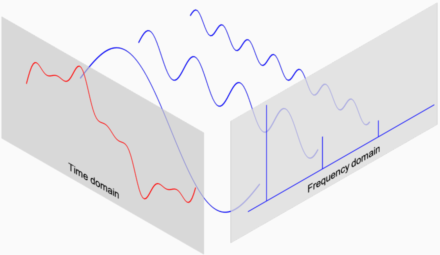
An example of a vector that represents a *noisy signal* whose characteristics we'd like to analyze.



The entries of the vector correspond to equally spaced time points and the value of the entry corresponds to the signal amplitude at that time.

The **Discrete Fourier Transform (DFT)** is a method to express every vector (i.e. every signal) as a linear combination of simple periodic vectors (i.e. complex sinusoidal signals).

Visually:



Discrete Fourier Transform

The **Discrete Fourier Transform (DFT)** is a method to express every vector (i.e. every signal) as a linear combination of simple periodic vectors (i.e. complex sinusoidal signals).

Mathematically: every vector $|\psi\rangle \in \mathbb{C}^N$ can be written as

$$|\psi\rangle = \sum_{j=0}^{N-1} \hat{\psi}_j \, |f_j\rangle$$

where $\{|f_0\rangle, |f_1\rangle, \ldots, |f_{N-1}\rangle\}$ is the $\mathbb{Z}_N$**-Fourier basis**.

Roots of unity

$$\omega_N = \exp\left(2\pi i/N\right)$$

are the N'th roots of unity. For example:

$$\omega_2 = -1 \qquad \text{and} \qquad \omega_2^2 = 1$$

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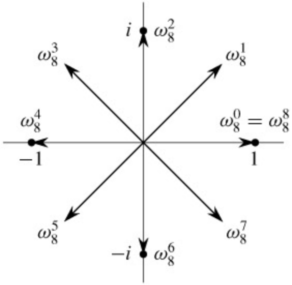
$$\omega_4 = i \qquad \omega_4^2 = -1 \qquad \omega_4^3 = -i \qquad \omega_4^4 = 1\ .$$

The eighth roots of unity drawn on the complex plane:



Roots of unity

Exponents of roots of unity follow **modular arithmetic**:

$$\omega_N^k = \omega_N^{k \mod N}.$$

For example, $\omega_8^{11} = \omega_8^3$ and $\omega_5^{-4} = \omega_5$. Why?

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They also satisfy the **Fourier identities**:

$$\sum_{j=0}^{N-1} \omega_N^{jk} = \begin{cases} 0 & k \neq 0 \mod N \\ N & k = 0 \mod N \end{cases}$$

Exercise: prove this!

Fourier Basis

The Fourier basis vectors are

$$|f_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{jk} \, |k\rangle$$

for $j = 0, 1, \ldots, N-1$.

Claim: The vectors $\{|f_j\rangle\}_j$ are unit vectors.

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Proof:

$$\| \, |f_j\rangle \, \|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\omega_N^{jk}|^2 = \frac{1}{N} \sum_{k=0}^{N-1} 1 = 1 \; .$$

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for $j = 0, 1, \ldots, N-1$.

Claim: The vectors $\{|f_j\rangle\}_j$ are pairwise orthogonal.

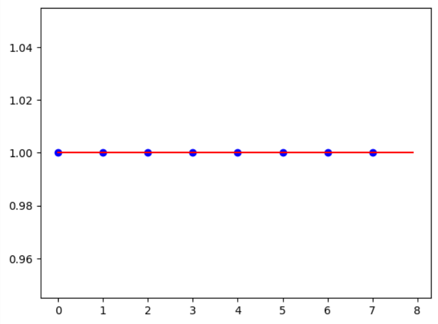
Proof: Fix $j \neq r$. Then

$$\begin{aligned}
\langle f_j \mid f_r\rangle &= \frac{1}{N}\Big(\sum_k \overline{\omega}_N^{jk} \, \langle k|\Big)\Big(\sum_\ell \omega_N^{\ell r} \, |\ell\rangle\Big) \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \omega_N^{-jk} \omega_N^{kr} && (\overline{\omega}_N^t = \omega_N^{-t}) \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \omega_N^{k(j-r)} = 0. && \text{(Fourier identity)}
\end{aligned}$$

Fourier Basis

Example: $N = 8, j = 0.$

$$|f_0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

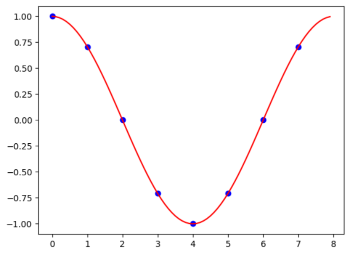


Fourier Basis

Example: $N = 8, j = 1$.

$$|f_1\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^k |k\rangle = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \omega_8 \\ \vdots \\ \omega_8^7 \end{pmatrix}$$

Visually, the real component of ω_N^{jk}:



Fourier Basis

Example: $N = 8, j = 5$.

$$|f_5\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{5k} |k\rangle = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \omega_8^5 \\ \omega_8^2 \\ \vdots \end{pmatrix}$$

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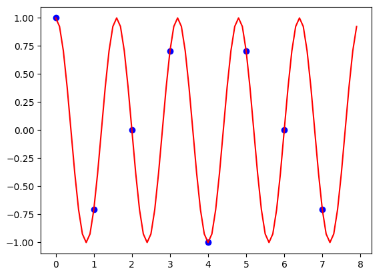

- Let $|\psi\rangle = \sum_{j=0}^{N-1} \psi_j |j\rangle$ be a signal represented in the **standard basis**.
- In the Fourier basis, $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{j=0}^{N-1} \hat{\psi}_j |f_j\rangle$ for some **Fourier coefficients** $\hat{\psi}_j$.

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- The magnitude $|\hat{\psi}_j|^2$ of the j'th Fourier coefficient quantifies the contribution of $|f_j\rangle$ to $|\psi\rangle$.
- What's the relationship between the amplitudes $\{\psi_0, \psi_1, \ldots, \psi_{N-1}\}$ and the Fourier coefficients $\{\hat{\psi}_0, \hat{\psi}_1, \ldots, \hat{\psi}_{N-1}\}$?

They are related by a *unitary* transformation F_N (which is the DFT). It maps $|j\rangle \mapsto |f_j\rangle$. The *inverse DFT* is $F_N^\dagger$, which maps $|f_j\rangle$ to $|j\rangle$.

Applying $F_N^\dagger$ to $|\psi\rangle$ yields the vector

$$F_N^\dagger |\psi\rangle = |\hat{\psi}\rangle = \sum_{j=0}^{N-1} \hat{\psi}_j |j\rangle .$$

The **Fourier coefficients** of $|\psi\rangle$ have been turned into *amplitudes* in the standard basis of $|\hat{\psi}\rangle$.