Week 4: Outer products, Teleportation, Measurements in other bases

COMS 4281 (Fall 2025)

Admin

1. Worksheet 3 available.
2. Pset1 released Wednesday evening, due October 8, 11:59pm.
3. Use EdStem to find pset collaborators. **However you must write your own solutions**.

Recap

1. Telling whether a state is entangled
2. No-cloning theorem
3. Partial measurements

Outer products

Outer products

Let $|a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle, |b\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$.

Inner product is a **number**:

$$\langle a|b\rangle = \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1$$

Outer product is a **matrix**:

$$|a\rangle \langle b| = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \begin{pmatrix} \overline{\beta}_0 & \overline{\beta}_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\overline{\beta}_0 & \alpha_0\overline{\beta}_1 \\ \alpha_1\overline{\beta}_0 & \alpha_1\overline{\beta}_1 \end{pmatrix}$$

Outer products

$$|a\rangle\langle b| = \begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix}\begin{pmatrix}\overline{\beta}_0 & \overline{\beta}_1\end{pmatrix} = \begin{pmatrix}\alpha_0\overline{\beta}_0 & \alpha_0\overline{\beta}_1 \\ \alpha_1\overline{\beta}_0 & \alpha_1\overline{\beta}_1\end{pmatrix}$$

Another perspective:

$$\begin{aligned} |a\rangle\langle b| &= \Big(\alpha_0|0\rangle + \alpha_1|1\rangle\Big)\Big(\overline{\beta}_0\langle 0| + \overline{\beta}_1\langle 1|\Big) \\ &= \alpha_0\overline{\beta}_0|0\rangle\langle 0| + \alpha_0\overline{\beta}_1|0\rangle\langle 1| + \alpha_1\overline{\beta}_0|1\rangle\langle 0| + \alpha_1\overline{\beta}_1|1\rangle\langle 1| \ . \end{aligned}$$

Notice that

$$|0\rangle\langle 0| = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\ , \quad |0\rangle\langle 1| = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\ , \quad |1\rangle\langle 1| = \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}$$

Outer products

Every matrix can be written as a linear combination of outer products. For example:

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1| \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1| \ .$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,|0\rangle\langle 0| + b\,|0\rangle\langle 1| + b\,|1\rangle\langle 0| + d\,|1\rangle\langle 1| \ .$$

Outer products

Matrix-vector multiplication using Dirac bra-ket notation. Let $|a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. Then

$$\begin{aligned}
\mathrm{I}\,|a\rangle &= \Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big)\Big(\alpha_0 |0\rangle + \alpha_1 |1\rangle \Big) \\
&= \alpha_0 |0\rangle\langle 0|0\rangle + \alpha_0 |1\rangle\langle 1|0\rangle + \alpha_1 |0\rangle\langle 0|1\rangle + \alpha_1 |1\rangle\langle 1|1\rangle \\
&= \alpha_0 |0\rangle + \alpha_1 |1\rangle \ .
\end{aligned}$$

as expected!

Outer products

The matrix $|a\rangle\langle a|$ is a **projection matrix**. It projects all vectors onto the direction of $|a\rangle$. Let $|b\rangle = \beta_0|0\rangle + \beta_1|1\rangle$. Then

$$\underbrace{|a\rangle\langle a|}_{\text{matrix}}\ \underbrace{|b\rangle}_{\text{vector}} = \underbrace{|a\rangle}_{\text{vector}}\ \underbrace{\langle a|b\rangle}_{\text{number}}$$
$$= \langle a|b\rangle\ |a\rangle\ .$$

Thus we get $|a\rangle$, scaled by the overlap between $|a\rangle$ and $|b\rangle$.

Outer products

Tensor product of outer products = outer products of tensor products.

$$|a\rangle\langle b|\otimes|c\rangle\langle d| = \Big(|a\rangle\otimes|c\rangle\Big)\Big(\langle b|\otimes\langle d|\Big) = |a,c\rangle\langle b,d| \; .$$

For example,

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = |0\rangle\langle 0|\otimes|0\rangle\langle 1| = |0,0\rangle\langle 0,1| = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}$$

You can check that both sides are equal to

$$\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Outer products

As usual with tensor products, match up the slots! Suppose $|\psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$.

Then

$$\Big(|a\rangle\langle b| \otimes |c\rangle\langle d| \Big)|\psi\rangle = \alpha\,|a\rangle\langle b|0\rangle \otimes |c\rangle\langle d|0\rangle + \beta\,|a\rangle\langle b|1\rangle \otimes |c\rangle\langle d|1\rangle$$

$$= \underbrace{\alpha\,\langle b|0\rangle\langle d|0\rangle}_{\text{number}}|a\rangle \otimes |c\rangle + \underbrace{\beta\,\langle b|1\rangle\langle d|1\rangle}_{\text{number}}|a\rangle \otimes |c\rangle$$

$$= \Big(\alpha\,\langle b|0\rangle\langle d|0\rangle + \beta\,\langle b|1\rangle\langle d|1\rangle\Big)|a\rangle \otimes |c\rangle \;\; .$$

Alternatively, you can interchange the outer product and tensor product operations to see that

$$\Big(|a\rangle\langle b| \otimes |c\rangle\langle d| \Big)|\psi\rangle = \Big(|a,c\rangle\langle b,d| \Big)|\psi\rangle$$

$$= \alpha\,|a,c\rangle\langle b,d|0,0\rangle + \beta\,|a,c\rangle\langle b,d|1,1\rangle$$

$$= \Big(\alpha\,\langle b,d|0,0\rangle + \beta\,\langle b,d|1,1\rangle\Big)|a,c\rangle$$

Quantum Teleportation

Quantum teleportation

Imagine Alice and Bob are friends, and Alice has a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that she wants to share with Bob. However Bob is really far away, and they can only communicate classically (so Alice can't just hand Bob her qubit). Can Alice transfer her exact qubit to Bob somehow?

She could try to call Bob over the phone and send a classical description of $|\psi\rangle$, but that could require an infinite number of bits if $|\psi\rangle$ has amplitudes that use transcendental numbers.

Quantum teleportation

If Alice and Bob preshare quantum entanglement, then Alice can **teleport** $|\psi\rangle$ to Bob.

Suppose one year ago, Alice and Bob were in the same quantum lab, and they generated the entangled state

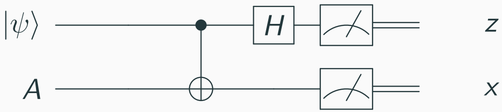
$$|\mathrm{EPR}\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) .$$

Alice takes the first qubit, Bob takes the second qubit and they go their separate ways.

Quantum teleportation

Fast forward to today, when Alice gets a gift qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

She performs the following circuit on her two qubits:



where A denotes Alice's half of the EPR pair, and the measurement outcomes are denoted $z, x \in \{0, 1\}$ respectively.

Quantum teleportation

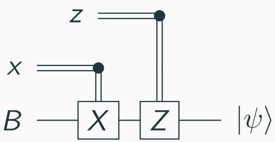
Alice calls up Bob over the phone: "I just teleported $|\psi\rangle$ over to you using the EPR pair we split a year ago."

Bob: "Nice! What corrections do I need to do?"

Alice tells Bob the bits z, x.

Quantum teleportation

Bob then applies the following gates to his EPR qubit B, depending on the values of x, z.

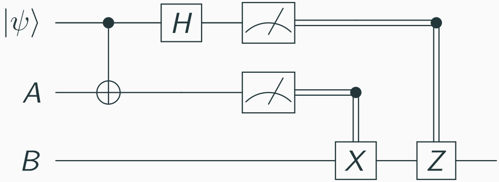


Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

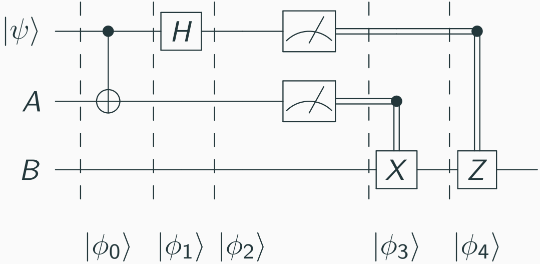
Quantum teleportation

Viewing the entire process as a circuit, this looks like



Quantum teleportation

We analyze it step by step:



Let's do this together on the board!

Quantum teleportation

Does this violate the No-Cloning Theorem?

No! Alice has measured her qubits, so no longer possesses $|\psi\rangle$.

Can this be used to send information faster than speed of light?

No! In order for Bob to recover $|\psi\rangle$, Alice needs to classically transmit the "correction bits" z, x to Bob, which is limited by the speed of light.

Both the preshared entanglement, and the classical communication are necessary for quantum teleportation to work.

Public Service Announcement

Entanglement does not allow for faster-than-light communication!