## Week 4: Nonstandard Measurements, Heisenberg Uncertainty Principle, EPR Paradox

COMS 4281 (Fall 2025)

#### **Admin**

- 1. Worksheet3 available, Quiz3 on Gradescope tonight.
- 2. Pset1 out (finally!), due October 10, 11:59pm.

## **AI** Policy

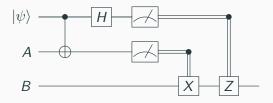
Syllabus has been updated with the Al policy. The gist of it is:

You may use AI/LLMs as a study aid, but not as a substitute for your own work. Quizzes and problem sets are practice for the midterm and final (70% of your grade), which you must do without any external help. Use AI wisely: ask it to explain, critique, or suggest alternatives — not just to give you the answer. Everything you submit must be something you understand and could explain to a peer without notes.

## Recap

- Outer products
- Quantum teleportation

Recall: the teleportation circuit.



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Both the preshared entanglement, and the classical communication are necessary for quantum teleportation to work.

#### **Public Service Announcement**

Entanglement does not allow for faster-than-light communication!

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This is what we call a **standard basis measurement** or a **computational basis measurement**: the outcomes are the standard basis (a.k.a. computational basis) vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \cdots \qquad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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Rewrite  $|\psi\rangle$  as a linear combination of the  $\{|+\rangle\,, |-\rangle\}$  basis:

$$|\psi\rangle = \alpha \frac{\sqrt{2}}{2} \Big( |+\rangle + |-\rangle \Big) + \beta \frac{\sqrt{2}}{2} \Big( |+\rangle - |-\rangle \Big)$$

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9

The probability of getting  $|+\rangle$  outcome is thus

$$\left(\frac{\sqrt{2}}{2}\right)^2 \cdot \left|\alpha + \beta\right|^2$$

and similarly the probability of  $\left|-\right\rangle$  outcome is

$$\left(\frac{\sqrt{2}}{2}\right)^2 \cdot \left|\alpha - \beta\right|^2$$

## **Another perspective**

Geometric perspective: measuring  $|\psi\rangle$  in diagonal basis is **projecting**  $|\psi\rangle$  along the  $|+\rangle\,, |-\rangle$  axes. The probability of either outcome is the square of the magnitude of the projections.

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Algebraically, we apply the projection matrices  $|+\rangle \langle +|$  and  $|-\rangle \langle -|$  to  $|\psi\rangle$ . Projection of  $|\psi\rangle$  along  $|+\rangle$  is

$$\left|v\right\rangle = \left|+\right\rangle\left\langle+\right|\left|\psi\right\rangle = \left\langle+\left|\psi\right\rangle\right.\left|+\right\rangle$$
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The probability of getting  $|+\rangle$  outcome is equal to

$$\left\| |\nu\rangle \right\|^2 = \left\| \langle +|\psi\rangle |+\rangle \right\|^2 = |\langle +|\psi\rangle|^2 \cdot \left\| |+\rangle \right\|^2 = |\langle +|\psi\rangle|^2$$

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Similarly probability of getting  $|-\rangle$  outcome is  $|\langle -|\psi\rangle|^2$ .

## General formula for measuring in a basis

Let  $|\psi\rangle\in\mathbb{C}^d$  be a quantum state. Let  $B=\{|b_1\rangle,\ldots,|b_d\rangle\}$  be an **orthonormal basis** for  $\mathbb{C}^d$ .

Measuring  $|\psi\rangle$  with respect to basis B yields outcome  $|b_j\rangle$  with probability

$$\left|\left\langle b_{j}\right|\psi\right\rangle \right|^{2}$$

and the post-measurement state is  $|b_j\rangle$ .

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Measuring  $|\psi\rangle$  in a basis B is **equivalent** to

- 1. Apply a unitary V that maps B to standard basis (i.e.  $|b_j\rangle \to |j\rangle$ ).
- 2. Measure  $V|\psi\rangle$  in standard basis to get outcome  $|j\rangle$ .
- 3. Apply inverse unitary  $V^{\dagger}$  to map  $|j\rangle$  back to  $|b_{j}\rangle$  as the post-measurement state.

**Example**: To measure in the diagonal basis, we have to perform a basis change from  $\{|+\rangle, |-\rangle\}$  to standard basis  $\{|0\rangle, |1\rangle\}$ . What unitary accomplishes this?

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After it collapses to  $|0\rangle$ , we apply  $H^{\dagger}=H$  to get  $|+\rangle$ , the post-measurement state.

#### Partial measurements plus nonstandard measurements

Let's combine the two concepts! Let  $|\psi\rangle=\sum_{i,j}\alpha_{ij}\,|i\rangle\otimes|j\rangle$  denote a two-qubit state. Say we measure the first qubit with respect to basis  $\{|b_0\rangle\,,|b_1\rangle\}$ .

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Measurement rule: Equivalent to applying the projections

$$|v_0\rangle = (|b_0\rangle \langle b_0| \otimes I) |\psi\rangle$$
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The probability of getting  $|b_0\rangle$  outcome is  $||v_0\rangle|^2$ , and the post-measurement state is  $|v_0\rangle/||v_0\rangle||$ .

Similarly, probability of getting  $|b_1\rangle$  outcome is  $||v_1\rangle||^2$ , and post-measurement state is  $|v_1\rangle/||v_1\rangle||$ .

Projecting  $|\psi\rangle$  along the first projection yields

$$\left( |b_0\rangle \langle b_0| \otimes \mathbf{I} \right) |\psi\rangle = \sum_{i,j} \alpha_{ij} |b_0\rangle \langle b_0|i\rangle \otimes \mathbf{I} |j\rangle$$

$$= \sum_{i,j} \alpha_{ij} \langle b_0|i\rangle |b_0\rangle \otimes |j\rangle$$

$$= |b_0\rangle \otimes \left( \sum_{i,j} \alpha_{ij} \langle b_0|i\rangle |j\rangle \right) .$$

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The probability of getting  $|b_0\rangle$  outcome is

$$\| |b_{0}\rangle \otimes \left( \sum_{i,j} \alpha_{ij} \langle b_{0} | i \rangle | j \rangle \right) \|^{2} = \underbrace{\| |b_{0}\rangle \|^{2}}_{1} \cdot \| \sum_{i,j} \alpha_{ij} \langle b_{0} | i \rangle | j \rangle \|^{2}$$

$$= \| \sum_{j} \left( \sum_{i} \alpha_{ij} \langle b_{0} | i \rangle \right) | j \rangle \|^{2}$$

$$= \sum_{i} \left| \sum_{i} \alpha_{ij} \langle b_{0} | i \rangle \right|^{2} .$$

Conditioned on the outcome  $|b_0\rangle$  in the first qubit, the post-measurement state of the two qubits is then

$$|b_{0}\rangle \otimes \frac{\sum_{i,j} \alpha_{ij} \langle b_{0} | i \rangle | j \rangle}{\left\| \sum_{i,j} \alpha_{ij} \langle b_{0} | i \rangle | j \rangle \right\|} = |b_{0}\rangle \otimes \frac{\sum_{i,j} \alpha_{ij} \langle b_{0} | i \rangle | j \rangle}{\sqrt{\sum_{j} \left| \sum_{i} \alpha_{ij} \langle b_{0} | i \rangle \right|^{2}}}$$

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Notice that, even though the original state  $|\psi\rangle$  could have been entangled, the post-measurement state is **unentangled**! Measurement generally destroy entanglement.

Let's work through an example:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$
.

Measure the **first** qubit in the **diagonal basis**.

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Measure the **first** qubit in the **diagonal basis**.

- What is the probability the outcome is  $|+\rangle$ ? Or  $|-\rangle$ ?
- What is the post-measurement state in either case?

#### Next time

 $\label{thm:principle} \mbox{Heisenberg's uncertainty principle, EPR paradox, Bell's Theorem.}$ 

Popular science version: can't exactly know both the position and momentum of a particle simultaneously.



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In other words, if measuring  $|\psi\rangle$  in standard basis yields a deterministic outcome, then it  ${\bf cannot}$  have a deterministic outcome if measured according to diagonal basis.

#### Quantitatively, let:

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Then a version of Heisenberg's uncertainty principle says

$$(p_0 - p_1)^2 + (p_+ - p_-)^2 \le 1$$

.

Important point: it's not about what happens if you sequentially measure the state  $|\psi\rangle$  (what happens then?).

**Important point**: it's not about what happens if you **sequentially** measure the state  $|\psi\rangle$  (what happens then?).

It's reasoning about **counterfactual scenarios**: measuring  $|\psi\rangle$  in the standard basis, **or** measuring  $|\psi\rangle$  in the diagonal basis.

We say that the standard basis and diagonal basis are incompatible or complementary.

In quantum physics, the position and momentum of a particle correspond to incompatible measurements!

# EPR Paradox



In 1935, Einstein, Podolsky, and Rosen published a paper called

Can Quantum-Mechanical Description of Physical Reality be

Considered Complete?

#### The EPR thesis:

Quantum mechanics may be very good at predicting outcomes of experiments, but it cannot be a **complete** description of Nature.

The reason they thought this was because of a thought experiment.

Alice and Bob are in far-away galaxies and share the EPR state

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$$|\Phi\rangle = \frac{1}{\sqrt{2}} \Big( |00\rangle + |11\rangle \Big).$$

Consider two possible experiments:

- Experiment A: Alice measures her qubit in the standard basis  $\{\ket{0},\ket{1}\}$
- $\bullet$  Experiment B: Alice measures her qubit in the diagonal basis  $\{|+\rangle\,, |-\rangle\}$

#### Alice gets outcome

- $|0\rangle$  with probability 1/2, and the post-measurement state is  $|00\rangle$ .
- $|1\rangle$  with probability 1/2, and the post-measurement state is  $|11\rangle$ .

To calculate the probability of getting outcome  $|+\rangle$ , we use the partial measurement + nonstandard basis rules: first, compute the vector

$$|\nu_{+}\rangle = \left( |+\rangle \langle +| \otimes I \right) |\Phi\rangle = |+\rangle \otimes \frac{1}{\sqrt{2}} \left( \langle +| 0\rangle |0\rangle + \langle +| 1\rangle |1\rangle \right)$$

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The squared length is  $\| |v_+\rangle \|^2 = \frac{1}{2}$ .

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Similarly, the probability of getting outcome  $|-\rangle$  is  $\frac{1}{2}$  and the post-measurement state is  $|-\rangle \otimes |-\rangle$ .

Alice and Bob are in far-away galaxies and share the EPR state  $|\Phi\rangle=\frac{1}{\sqrt{2}}\Big(\left|00\right\rangle+\left|11\right\rangle\Big).$  If Alice measures in standard basis, after seeing her result she knows exactly what state Bob's qubit is in – even if Bob's qubit is zillions of lightyears away.

Einstein's (and Podolsky's and Rosen's) reasoning:

1. If Alice did Experiment A and got outcome (say)  $|0\rangle$ , then it must have been Bob's qubit was really  $|0\rangle$  all along.

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- 3. But Alice's choice of measurement (standard or diagonal) couldn't have made an instantaneous difference in intrinsic the state of Bob's qubit, right?
- 4. Therefore Bob's qubit must have answers prepared for both Experiments simultaneously – violating Heisenberg's Uncertainty Principle!

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EPR's thesis: There is a **deeper** classical theory – called a **local** hiden variable theory – that

- Reproduces the same statistics as Quantum Mechanics
- But has hidden variables that describes the intrinsic state of particles.
- Respects the speed of light limit.

What do you think of Einstein's reasoning? It is a paradox?