Quantum Information Basics

COMS 4281 (Fall 2025)

Admin

1. Pset0, Quiz1 due Friday Sept 12, 11:59pm.
2. Pset1 out this weekend.
3. Worksheet 1 out. Attend office hours!

Weekly quizzes

- On most weeks, there will be a Gradescope quiz to help you follow the class material. Released Monday morning, and must be completed by the following Sunday night.
- Doable in ~ 15 minutes if you understand the class material to date.
- The quiz will be based on a weekly worksheet to help you practice. The TAs will go over the worksheet in office hours.
- **Questions on the midterm/final will also be based on the worksheets**.

Last Time: classical reversible computing

d-dimensional systems:

- State labels: $|0\rangle, \ldots, |d-1\rangle$.
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Can implement universal classical computation.

Last Time: classical reversible computing, linear algebra-ized

States represented as column vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} \qquad \cdots \qquad |d-1\rangle = \begin{pmatrix} \vdots \\ 0 \\ 1 \end{pmatrix}$$

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Transformations are $d \times d$ permutation matrices, e.g.,

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Last Time: classical reversible computing, linear algebra-ized

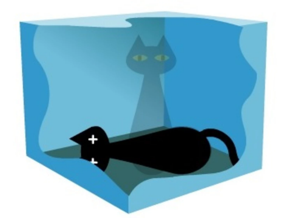
Updating a state $|x\rangle$ by transformation T is matrix-vector multiplication $T|x\rangle$.

Tensor product of vectors and matrices corresponds to combining states and transformations

Making the quantum leap

A **bit** is a classical system with *two* distinguishable states $|0\rangle, |1\rangle$, also called *Classical states*, or *standard basis states*.

A **qubit** (quantum bit) can be in a **superposition** of the classical states $|0\rangle, |1\rangle$.



Mathematically, states of a qubit are **complex linear combination** of $|0\rangle, |1\rangle$:

$$
\begin{aligned}
|\psi\rangle &= \alpha |0\rangle + \beta |1\rangle \\
&= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\end{aligned}
$$

where $\alpha, \beta \in \mathbb{C}$ are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

In other words, $|\psi\rangle$ is a two-dimensional unit vector in $\mathbb{C}^2$.

Example: a qubit can be in the state

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle .$$

Another example:

$$\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{i}{\sqrt{2}}\left|1\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

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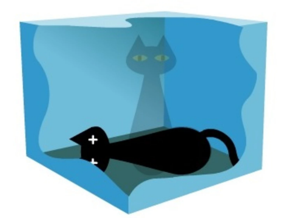
A non-valid qubit state:

$$i|0\rangle - \frac{1}{2}|1\rangle.$$

So what is a qubit

A qubit in the state $\alpha |0\rangle + \beta |1\rangle$ is commonly said to be $|0\rangle$ and $|1\rangle$ "at the same time". But what does that mean?



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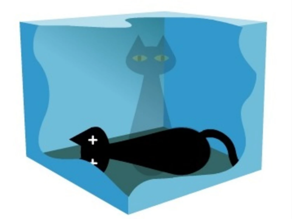

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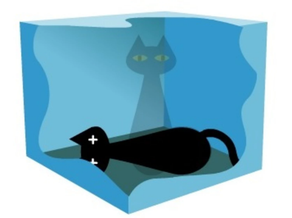

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α, β are called the **amplitudes** of the states $|0\rangle$ and $|1\rangle$, respectively.

Observing qubits

The state of a qubit cannot be directly observed. It must be **measured**, yielding a classical state $|0\rangle$ or $|1\rangle$ with probabilities

$$\Pr[\text{ observing } |0\rangle\ ] = |\alpha|^2 \qquad \Pr[\text{ observing } |1\rangle\ ] = |\beta|^2.$$

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Because qubit states have unit length, these probabilities add up to 1.

This formula is called the **Born Rule**.

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After measurement, the system becomes **classical**.

The state of qubit **collapses** to either $|0\rangle$ or $|1\rangle$, and the previous state is lost.

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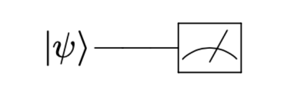
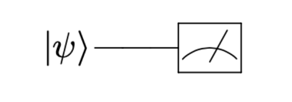
We represent qubit measurements using this diagram:

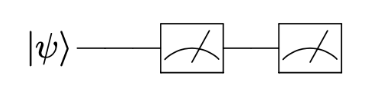


Observing qubits

If the state collapses to the classical state $|0\rangle$ and we measure it again, it stays in state $|0\rangle$ **with probability** 1. Same with collapsing to $|1\rangle$.

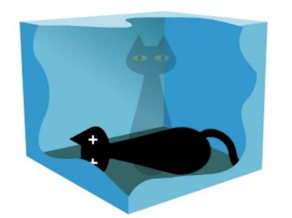


same as



Measuring a system twice is the same as measuring once.

Example: Schrödinger's cat



A box with two classical states:

sleeping cat: $|0\rangle$ awake cat: $|1\rangle$

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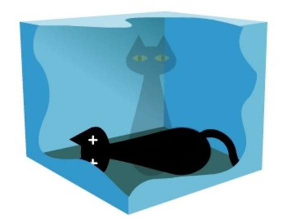
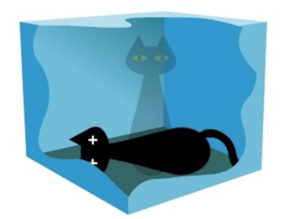


A box with two classical states:

sleeping cat: $|0\rangle$ awake cat: $|1\rangle$

In quantum mechanics, the box can be in a superposition of sleeping and awake cat, *as long as you don't open the box* (i.e. measure it).

Example: Schrödinger's cat



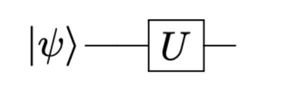
Suppose the box starts in the state $\frac{1}{\sqrt{3}}|0\rangle - i\frac{\sqrt{2}}{\sqrt{3}}|1\rangle$ and is measured.

1. With probability $\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$, the state collapses to $|0\rangle$ (i.e., sleeping cat).
2. With probability $\left|-i\frac{\sqrt{2}}{\sqrt{3}}\right|^2 = \frac{2}{3}$, the state collapses to $|1\rangle$ (i.e., awake cat).

Transformations on qubits

In addition to measurement, the state of a qubit can change via a **unitary transformation**. Just like transformations in classical reversible computing, unitary tranformations can be represented as matrices.

We represent a unitary transform U acting on state $|\psi\rangle$ using the following circuit diagram:



Several equivalent definitions of unitary matrices

Definition 1. The inverse of U is its Hermitian conjugate $U^\dagger$, pronounced "U dagger", whose (i,j)'th entry is the *complex conjugate* of the (j,i)'th entry of U:

$$U^\dagger_{i,j} = \overline{U}_{j,i}$$

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Definition 3. The rows of U form an orthonormal basis for $\mathbb{C}^d$, and the columns form an orthonormal basis for $\mathbb{C}^d$.

Definition 4. U preserves the inner products between vectors: inner product between $|\psi\rangle$ and $|\theta\rangle$ is the same as the inner product between $U|\psi\rangle$ and $U|\theta\rangle$.

Examples of qubit unitary matrices

Identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

For all qubit states $|\psi\rangle$, $I\,|\psi\rangle = |\psi\rangle$.

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Bit flip matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha X|0\rangle + \beta X|1\rangle = \alpha|1\rangle + \beta|0\rangle \; .$$

So far, have only seen classical transformations.

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Phase flip matrix $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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Examples of qubit unitary matrices

Hadamard matrix $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H\left|0\right\rangle = \cdots \text{(do on board)} \cdots$$

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$$H|0\rangle = \cdots \text{(do on board)} \cdots$$

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H maps classical basis states $|0\rangle, |1\rangle$ into **quantum** superpositions.

Quantum vs classical bits

What is the difference between

$|+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle \Big)$ and

$|-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle \Big)$?

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Measuring both states yields the same statistical outcomes:

$|0\rangle, |1\rangle$ with 50% probability each!

Quantum vs classical bits

Suppose we were physically handed a qubit (say Schrödinger's box) whose state $|\psi\rangle$ was either $|+\rangle$ or $|-\rangle$. Is there a way we can tell the difference?

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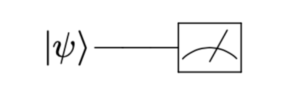
Suppose we were physically handed a qubit (say Schrödinger's box) whose state $|\psi\rangle$ was either $|+\rangle$ or $|-\rangle$. Is there a way we can tell the difference?

Opening the box (i.e., measuring) would yield a sleeping or awake cat with equal probability in both cases.



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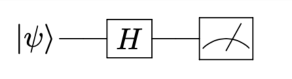

Solution: Apply H to qubit before measuring!

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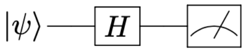


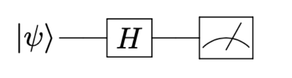
Solution: Apply H to qubit before measuring!

Case 1: $|\psi\rangle = |+\rangle$. Applying H, we get

$$H|+\rangle = \cdots \text{(show on the board)} \cdots = |0\rangle$$

Measuring yields $|0\rangle$ all the time!

Quantum vs classical bits



Solution: Apply H to qubit before measuring!

Case 2: $|\psi\rangle = |-\rangle$. Applying H, we get

$$H\,|-\rangle = \cdots \text{(show on the board)} \cdots = |1\rangle$$

Measuring yields $|1\rangle$ all the time!

Quantum vs classical bits

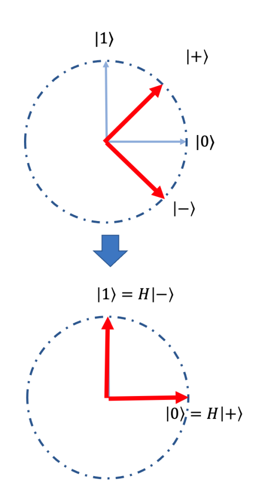
The states

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

and

$$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

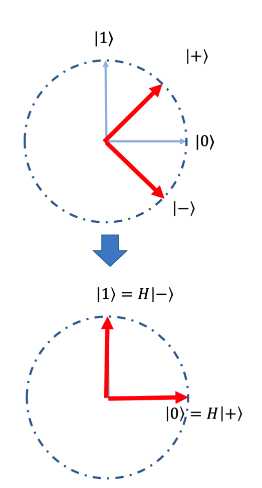
form an orthonormal basis for $\mathbb{C}^2$.



Quantum vs classical bits

In quantum mechanics, orthogonal states can be **perfectly distinguished** by applying an appropriate unitary matrix and then measuring in the standard basis.

The Hadamard matrix maps the $\{ |+\rangle, |-\rangle \}$ basis to the standard $\{ |0\rangle, |1\rangle \}$ basis (and vice versa).



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More precisely, **relative phases** between the classical basis states matter.

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More precisely, **relative phases** between the classical basis states matter.

On the other hand, **global phases** don't matter.

There is no quantum process (unitary + measurement) to distinguish between $|\psi\rangle$ and $-|\psi\rangle$, or in fact $\alpha|\psi\rangle$ for any complex phase $\alpha = e^{i\theta}$. Can you see why?