COMS 4281 - Introduction to Quantum Computing Fall 2025

Practice Worksheet 2 - More Quantum Math

This practice worksheet is intended to cover material up to September 9, which includes the basics of quantum math. The weekly quiz (due September 19, 11:59pm) will be based on this worksheet. The midterm and final exam will have questions inspired by the worksheets.

Last week, we saw that there are two types of notation:

- *Ket notation*: This is when we write $|\psi\rangle, |x\rangle, |0\rangle$, etc to denote quantum states.
- *Matrix/vector notation*: This is when we write a quantum state or a quantum transformation as an explicit vector or matrix, e.g. we write that $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the first few weeks of this class, we will jump back and forth very often between these two notation systems. As the class goes on, however, we will almost always use ket notation and only use matrix/vector notation when we're talking about a few qubits. That's because for 3 or more qubits, it gets unwieldy to write out all the entries of the corresponding matrix/vector.

First, let's make sure we understand how to translate from ket notation to vector notation, and vice versa. Recall the "standard basis", or "classical" states of a qubit:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Problem 1: From kets to vectors, and back

(a) Consider the following two states, which together form the "Hadamard basis" for a qubit:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

What are the vectors corresponding to these $|+\rangle, |-\rangle$?

(b) Express $|0\rangle$ and $|1\rangle$ as linear combinations of $|+\rangle$ and $|-\rangle$. That is, write the standard basis vectors in the Hadamard basis.

(c) Consider the following vector:

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -i\sqrt{\frac{2}{3}} \end{pmatrix} .$$

Write $|\psi\rangle$ as a linear combination of $|0\rangle, |1\rangle$. In other words, express $|\psi\rangle$ in terms of the standard basis.

(d) Write $|\psi\rangle$ as a linear combination of $|+\rangle, |-\rangle$. In other words, express $|\psi\rangle$ in terms of the Hadamard basis.

In class we discussed how we use the tensor product notation to describe the state of a composite system (e.g. a system consisting of multiple qubits). For example, $|0\rangle \otimes |1\rangle$ describes the state of two qubits where the first is in state $|0\rangle$, and the second is in state $|1\rangle$. When we translate to vector notation, however, we have to adhere to a convention. There are many possible conventions, but the one we will use throughout class is the one where $|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, and so on.

Similarly, we call $|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle$ the "standard basis" for two qubits. Furthermore we use the following abbreviations:

$$|0\rangle \otimes |0\rangle = |0\rangle |0\rangle = |0,0\rangle = |00\rangle \ .$$

Problem 2: Tensor products

(a) Express $|+\rangle \otimes |-\rangle$ as a linear combination of the standard basis states (using *ket* notation) for two qubits.

(b) Express $|+\rangle \otimes |-\rangle$ in vector notation.

(c) Express the following vector $|\psi\rangle$ in terms of the standard basis kets:

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix} \ .$$

Problem 3: Tensor Products and Gates

Here we're going to get some practice applying gates to multiple qubits both using the tensor notation and the matrix/vector notation. We're going to see that these two notations are consistent with each other.

(a) Simplify, using ket notation, the expression $(X \otimes H)(|0\rangle \otimes |+\rangle)$. Try to do this without using any vector notation.

(b) Write the 4×4 matrix corresponding to the gate $X \otimes H$.

(c) Write the 4-dimensional vector corresponding to $|0\rangle \otimes |+\rangle$.

(d) Write out the 4-dimensional vector corresponding to $|-\rangle \otimes |0\rangle$.

Problem 4: Linear operators

Unitary quantum operators can be understood as "linear maps" on quantum states of some number of qubits. Linear maps, in turn, are completely specified by how they transform basis elements. For example, to define an arbitrary transformation U on a single qubit, we can specify $U|0\rangle = |+\rangle$ and $U|1\rangle = |-\rangle$. Those two constraints uniquely constrain U, and in this case U will be the Hadamard matrix.

Consider the following linear map $A : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, that acts on the standard basis as follows:

$$\begin{aligned} A\,|00\rangle &= |10\rangle \\ A\,|01\rangle &= |11\rangle \\ A\,|10\rangle &= |01\rangle \\ A\,|11\rangle &= |00\rangle \ . \end{aligned}$$

(a) Write A as a 4×4 matrix.

(b) Is A unitary?

(c) What is A^{-1}, i.e., its inverse, as a matrix? How does the inverse behave on standard basis states?

Problem 5: Basic circuits

(a) Write out the states $|\psi_n\rangle$ (shown in the circuit diagram in figure 1) in ket notation, for $n = 0, 1, 2, 3$.

(b) Write $|\psi_3\rangle$ in both the standard and Hadamard bases.

(c) What are the measurement statistics of measuring $|\psi_3\rangle$?

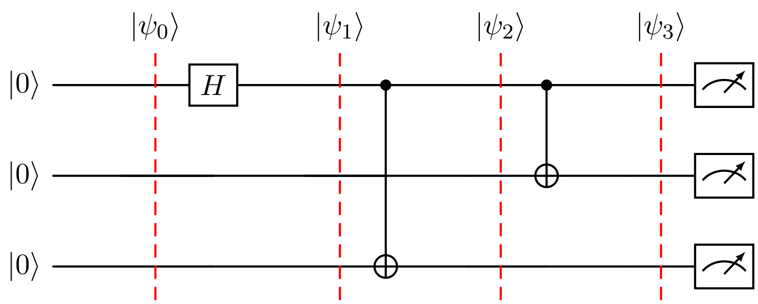

Figure 1: What are all the intermediate states?

Problem 6: Extracting measurement statistics

Throughout, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

(a) **Global phase.** Prove (informally) that multiplying a quantum state by a global phase cannot change measurement probabilities. In other words, show that for any quantum state $|\psi\rangle = \sum_j c_j\,|j\rangle$ and any real number α, the state $e^{i\alpha}\,|\psi\rangle$ has the same measurement probabilities as $|\psi\rangle$.

(b) **Relative phase.** Let $|\psi_\varphi\rangle = (|0\rangle + e^{i\varphi}\,|1\rangle)/\sqrt{2}$.

i. If you measure directly in the computational basis, what are the outcome probabilities?

ii. If you first apply a Hadamard gate and then measure in the computational basis, what are the outcome probabilities? For which values of φ is the result deterministic?