COMS 4281 - Introduction to Quantum Computing Fall 2025

Practice Worksheet 3 - Inner Products, Entanglement, Measurement

This practice worksheet is intended to cover material up to September 18th. The weekly quiz (due September 26th, 11:59pm) will be based on this worksheet. The midterm and final exam will have questions inspired by the worksheets.

Last week, we learned about inner products, entanglement, the No-Cloning Theorem, and partial measurements.

Problem 1: Inner products

Simplify the following expressions. Recall that, given two states $|\psi\rangle$ and $|\theta\rangle$, their inner product $\langle\psi|\theta\rangle$ is a scalar.

(a) $\langle 1|0\rangle$

(b) $\langle -|+\rangle$

(c) $\langle +|1\rangle$

(d) $\langle 0,1|+,-\rangle$.

Problem 2: Outer products and operators

Recall that, given two states $|\psi\rangle$ and $|\theta\rangle$. their outer product $|\psi\rangle\langle\theta|$ is a matrix. For (a)–(c), write the matrix form of the operator and identify its commonly-known name.

(a) $|1\rangle\langle 0| + |0\rangle\langle 1|$

(b) $\sqrt{2}(|+\rangle\langle -| + |0\rangle\langle 1|)$

(c) $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$.

(d) Let U be an arbitrary single-qubit gate. Describe the high-level functionality of the matrix

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$$

as a quantum gate. We often call this gate the “controlled-U” gate, or abbreviated by CU.

Problem 3: Entanglement

Recall that a two-qubit state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is *unentangled* (or, equivalently, a *product state*) if it can be written in the form of $|\varphi\rangle \otimes |\theta\rangle$ where $|\varphi\rangle, |\theta\rangle \in \mathbb{C}^2$ are single-qubit states. Otherwise, we call it *entangled*.

For the following states, show which ones are entangled, and which ones are not.

(a) $\frac{1}{2}\Big(|00\rangle - |01\rangle + |10\rangle + |11\rangle\Big)$

(b) $\frac{1}{2}\Big(|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big)$

For states that have three or more qubits, entanglement gets more subtle. We say that an n-qubit state $|\psi\rangle$ is *completely unentangled* if it can be written as $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ where each of the $|\psi_i\rangle$'s are single-qubit states. Even if it's not completely unentangled, different groups of qubits can entangled with each other, and unentangled between groups. For example, in a 3-qubit state, the 1st qubit could be unentangled with the 2nd and 3rd, but those two are entangled with each other. States that only have one group of entangled qubits are called *completely entangled*.

In the following three qubit states, identify the groupings of qubits that entangled with each other, and unentangled between groups. Which states are completely unentangled, and which ones are entangled?

(a) $\frac{1}{\sqrt{8}}\Big(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle \Big)$

(b) $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle \Big)$

(c) $\frac{1}{\sqrt{2}}\Big(|000\rangle + |110\rangle \Big)$

(d) $\frac{1}{2}\Big(|000\rangle + |010\rangle - |101\rangle - |111\rangle \Big)$

Problem 4: Partial Measurements

Consider performing a partial (standard basis) measurement on the first qubit of each of the following states. For each of the outcomes (either $|0\rangle$ or $|1\rangle$), what is (i) the unnormalized state after the partial measurement, (ii) the probability of that outcome, and (iii), the post-measurement state?

(a) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

(b) $|\psi\rangle = \frac{1}{\sqrt{5}}|01\rangle + \sqrt{\frac{2}{5}}|10\rangle - \sqrt{\frac{2}{5}}|11\rangle$

(c) $|\psi\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|100\rangle + \frac{1}{2}|110\rangle$. Note that this is a three-qubit state. *Hint*: Before computing the partial measurement, what can you first observe about the state?

Now do (a)-(c) again, except the partial measurement is in the Hadamard basis $\{|+\rangle, |-\rangle\}$.