COMS 4281 - Introduction to Quantum Computing Fall 2025

Practice Worksheet 4 - Outer Products, Teleportation, Uncertainty

This practice worksheet is intended to cover material up to September 27th. The weekly quiz (due October 3rd, 11:59pm) will be based on this worksheet. The midterm and final exam will have questions inspired by the worksheets.

Last week, we learned about outer products, teleportation, EPR paradoxes and uncertainty.

Problem 1: Dirac notation level 2

Here we will exercise our comfort levels with the Dirac notation.

Simplify the following quantities. Remember, match up the slots correctly!

(a)

$$\Big(\left|+\right\rangle \left\langle+\right| \otimes \left|-\right\rangle \left\langle-\right| \Big)\Big(\left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle - \left|11\right\rangle \Big) .$$

(b)

$$\Big(\left|+\right\rangle \left\langle+\right| \ + \ \left|-\right\rangle \left\langle-\right| \Big)\Big(\alpha \left|0\right\rangle + \beta \left|1\right\rangle \Big) .$$

What is the result in the standard basis? What do you conclude about the matrix in the first parenthesis?

(c)

$$\Big(\mathrm{I} \otimes \left|0\right\rangle \left\langle-\right| \Big) \left|+\right\rangle \otimes \left|-\right\rangle$$

Let $\left|a\right\rangle = \sum_{j=1}^{d} \alpha_j \left|j\right\rangle, \left|b\right\rangle = \sum_{j=1}^{d} \beta_j \left|j\right\rangle$ be vectors in $\mathbb{C}^d$ (not necessarily normalized).

(a) Write the matrix representation of the outer product $\left|a\right\rangle \left\langle b\right|$.

(b) Prove that

$$\Big(\left|a\right\rangle \left\langle b\right| \Big)^{\dagger} = \left|b\right\rangle \left\langle a\right| \ .$$

In other words, the Hermitian conjugate (i.e., complex conjugate followed by transpose) flips the outer product.

(c) Using Dirac notation only, prove that when $\left|a\right\rangle$ has unit length the matrix

$$R = \mathrm{I} - 2 \left|a\right\rangle \left\langle a\right|$$

is a unitary matrix. *Hint*: which one of the four definitions of unitary matrix given in class is appropriate?

(d) What does the R matrix do (i.e., when multiplied with) to vector $\left|a\right\rangle$? What does it do to vectors $\left|c\right\rangle$ that are orthogonal to $\left|a\right\rangle$? Again, only use bra-ket notation.

(e) Based on your answers above, describe its behavior on arbitrary vectors $\left|\psi\right\rangle \in \mathbb{C}^d$. *Hint*: express $\left|\psi\right\rangle$ as a linear combination of $\left|a\right\rangle$ and other vectors.

Problem 2: EPR pair

Recall our favorite entangled state:

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \ .$$

Let $\{|a\rangle, |b\rangle\}$ be orthonormal basis for $\mathbb{C}^d$ such that, when written as column vectors, have only *real-valued* entries (i.e., no imaginary components). Prove that we can also write our trusty EPR pair as

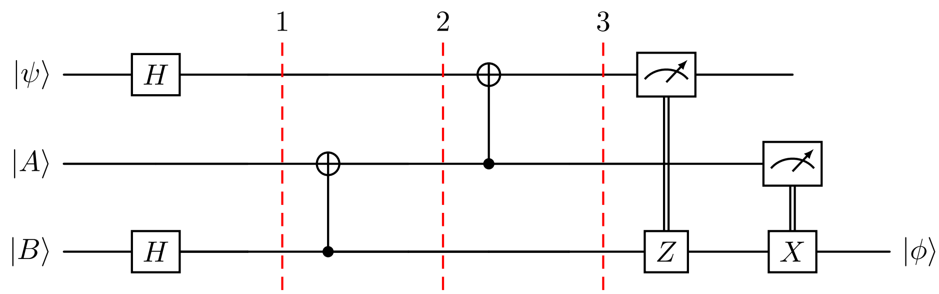
$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}}|aa\rangle + \frac{1}{\sqrt{2}}|bb\rangle \ .$$

In other words, the EPR pair is maximally entangled in *every* (real-valued) basis!

Hint: You really have to use the orthogonality of $|a\rangle, |b\rangle$ plus the fact they are unit norm!

Problem 3: Teleportation

In class we saw that if Alice and Bob shared entanglement that given another qubit in a arbitrary state Alice could teleport it. Here's Alice and Bob are exploring different circuit options for the same desired teleported state result. Assume that in this case $|A\rangle = |B\rangle = |0\rangle$ at the start.



(a) What is the overall state of the system at line 1?

(b) What is the overall state of the system at line 2?

(c) What is the overall state of the system at line 3?

(d) What's the key difference in the way the $|\phi_2\rangle$ statevector from class can be grouped?

(e) Prove or disprove $|\psi\rangle = |\phi\rangle$ for some arbitrary $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Did this alternate circuit teleport $|\psi\rangle$? Why?

Problem 4: Uncertainty

Recall from class (slide 22): $p_0 = |\langle 0|\psi\rangle|^2$, $p_1 = |\langle 1|\psi\rangle|^2$, $p_+ = |\langle +|\psi\rangle|^2$, and $p_- = |\langle -|\psi\rangle|^2$. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

(a) Compute $p_0 - p_1$ and $p_+ - p_-$ in terms of α and β.

(b) Show the given version of Heisenberg's uncertainty principle holds for any normalized qubit state. In this part assume $\alpha, \beta \in \mathbb{R}$ to simplify things.