COMS 4281 - Introduction to Quantum Computing Fall 2025

Practice Worksheet 7 - Quantum counting and error correction

This practice worksheet is intended to cover material up to November 18. The weekly quiz (due November 21th, 11:59pm) will be based on this worksheet. The final exam will have questions inspired by the worksheets.

Problem 1: Quantum Counting

Let $f : \{0,1\}^n \to \{0,1\}$ denote a function with $M \leq N$ solutions. Consider one Grover iteration $G = RO_f$ where $R = 2|s\rangle\langle s| - \mathrm{I}$ is the Grover diffusion operator and O_f is the phase oracle corresponding to f.

(a) Let $|\Gamma\rangle, |\Delta\rangle$ denote the uniform superpositions over non-solutions and solutions, respectively. Show that

$$G|\Gamma\rangle = \cos 2\theta\,|\Gamma\rangle + \sin 2\theta\,|\Delta\rangle$$
$$G|\Delta\rangle = -\sin 2\theta\,|\Gamma\rangle + \cos 2\theta\,|\Delta\rangle$$

where $\sin\theta = \sqrt{M/N}$. In other words, on the two-dimensional subspace spanned by $|\Gamma\rangle, |\Delta\rangle$, the Grover iterate G acts as the 2×2 matrix

$$M = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} .$$

(b) Show that M has eigenvectors

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\Big(|\Gamma\rangle \pm i\,|\Delta\rangle \Big)$$

with corresponding eigenvalues $e^{-i2\theta}$ and $e^{i2\theta}$ respectively.

Problem 2: Classical error-correction

Consider the repetition code where 1 logical bit is encoded into k physical bits (assume that k is odd):

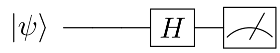
$$0 \mapsto \underbrace{00\cdots0}_{k}$$
$$1 \mapsto \underbrace{11\cdots1}_{k} .$$

Suppose I encode a single bit b using this repetition code and send it across a noisy channel. How many bit flip errors can this encoding tolerate in order for the receiver to unambiguously determine what my original bit b was?

Problem 3: Measurement vs. entanglement

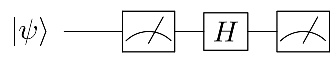
In this problem you will explore the connection between *measurement* and *entanglement with the environment*. Mathematically, these are two different ways of describing the same thing.

Consider the following circuit, which is part of something you might find in, e.g., a phase estimation procedure: let $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{2\pi i\theta}|1\rangle\Big)$.



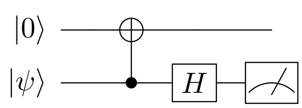
(a) What is the distribution of outcomes in the circuit above?

Now, let's consider inserting an intermediate measurement:



(b) What is the distribution of outcomes in the circuit above? Does this distribution depend on θ in any way?

Now let's consider yet another circuit:



(c) What is the distribution of outcomes in the circuit above?
Hint: use the partial measurement rules we learned about way back when.

If we think of this extra qubit starting in the $|0\rangle$ qubit as part of the environment, then the CNOT can be viewed as an unwanted interaction between $|\psi\rangle$ and the environment. This has the same effect as measurement.

Problem 4: The phase-flip code

Consider the following unitary matrix

$$E = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\theta} \end{pmatrix} .$$

This is a general phase gate. Suppose that $\theta \ll 1$, and imagine that E is due to some noise in the quantum hardware.

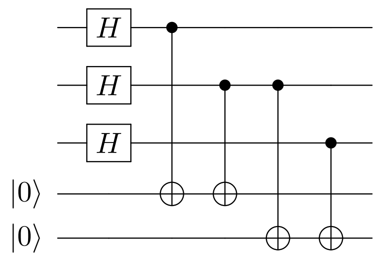
(a) Show that E can be expressed as a complex linear combination of I, X, Z, XZ where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(b) Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Suppose we encode $|\psi\rangle$ using the 3-*qubit phase-flip code* covered in class. What is the resulting encoded state $|\overline{\psi}\rangle$?

(c) Let $\left|\widetilde{\psi}_1\right\rangle = E_1 |\overline{\psi}\rangle$. In other words, we apply the error E to the first qubit of the encoded state. What is the resulting state?

(d) Now suppose we run the following phase-flip syndrome measurement circuit on the noisy state $\left|\widetilde{\psi}_1\right\rangle$, where the last two qubits are syndrome qubits:



What is the global state (denote this by $|\varphi\rangle$), including the syndrome qubits?

(e) Suppose we measure the syndrome qubits of $|\varphi\rangle$. What is the distribution of outcomes? What are the post-measurement states?

Problem 5: Shor code error-correction capability

In class, we argued that Shor's 9-qubit code can correct any single-qubit error. It can in fact handle a slightly broader class of errors; it can handle two-qubit errors as long as one of the qubits was affected by a bit flip, and the other was affected by a phase flip.

Give a brief (no more than 2-3 sentences) explanation for why this is the case.