

Composite Systems and Partial Measurements

Lecture 4: Tensor products and entanglement

COMS 4281: Introduction to Quantum Computing

Last Time: qubits and measurement

Qubit states and phase

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

A global phase does not change the physical state; relative phase matters.

Measurement depends on the basis

$$\Pr[\text{outcome } b] = |\langle v_b | \psi \rangle|^2, \quad \text{post-measurement state: } |v_b\rangle.$$

Here $\{|v_0\rangle, |v_1\rangle\}$ is an orthonormal basis.

Unitary evolution changes amplitudes without measuring:

$$|\psi\rangle \mapsto U |\psi\rangle, \quad H |+\rangle = |0\rangle, \quad H |-\rangle = |1\rangle.$$

Last Time: more than one qubit

- **Dirac notation:** bras, kets, and inner products.
- **Composite systems:** tensor products and multi-qubit states.
- **Entanglement:** Bell pairs, and how CNOT can create them.
- **Partial measurements:** outcomes and conditional states.

Today: consolidate these ideas through worked examples.

Composite Systems

Composite quantum systems

Suppose qubit A is prepared in state $|\psi_1\rangle$ and qubit B is prepared independently in state $|\psi_2\rangle$.

Their joint state is the **tensor product**

$$|\psi_1\rangle_A \otimes |\psi_2\rangle_B.$$

Composite quantum systems

Suppose qubit A is prepared in state $|\psi_1\rangle$ and qubit B is prepared independently in state $|\psi_2\rangle$.

Their joint state is the **tensor product**

$$|\psi_1\rangle_A \otimes |\psi_2\rangle_B.$$

This is a **four-dimensional** unit vector in

$$\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2.$$

We call such a state a **product state**.

Hilbert spaces and tensor products

For this course, a finite-dimensional **Hilbert space** is a space like $\mathbb{C}^d$, equipped with an inner product.

Suppose $\mathcal{H}_A$ and $\mathcal{H}_B$ have orthonormal bases

$$\{|a_1\rangle, \dots, |a_m\rangle\}, \quad \{|b_1\rangle, \dots, |b_n\rangle\}.$$

Hilbert spaces and tensor products

For this course, a finite-dimensional **Hilbert space** is a space like $\mathbb{C}^d$, equipped with an inner product.

Suppose $\mathcal{H}_A$ and $\mathcal{H}_B$ have orthonormal bases

$$\{|a_1\rangle, \dots, |a_m\rangle\}, \quad \{|b_1\rangle, \dots, |b_n\rangle\}.$$

Then $\mathcal{H}_A \otimes \mathcal{H}_B$ has the orthonormal basis

$$\{|a_i\rangle \otimes |b_j\rangle : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Dimensions multiply:

$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = mn.$$

Two qubits: basis and ordering

The computational basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle.$$

$$|ij\rangle = |i\rangle_A \otimes |j\rangle_B.$$

Two qubits: basis and ordering

The computational basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle.$$

$$|ij\rangle = |i\rangle_A \otimes |j\rangle_B.$$

In this order, the basis vectors are

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Convention: the first bit refers to A ; the second refers to B . In a circuit, qubits are ordered top to bottom.

Tensor products: distribute the amplitudes

Let

$$|\psi_1\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\psi_2\rangle = \gamma |0\rangle + \delta |1\rangle.$$

Tensor products: distribute the amplitudes

Let

$$|\psi_1\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\psi_2\rangle = \gamma |0\rangle + \delta |1\rangle.$$

Then

$$\begin{aligned} |\psi_1\rangle \otimes |\psi_2\rangle &= (\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle) \\ &= \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle. \end{aligned}$$

Tensor products: distribute the amplitudes

Let

$$|\psi_1\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\psi_2\rangle = \gamma |0\rangle + \delta |1\rangle.$$

Then

$$\begin{aligned} |\psi_1\rangle \otimes |\psi_2\rangle &= (\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle) \\ &= \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle. \end{aligned}$$

Equivalently,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}.$$

Examples of product states

Example 1

$$|0\rangle \otimes |1\rangle = |01\rangle .$$

Example 2: expand this tensor product.

$$|+\rangle \otimes |-\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle).$$

Examples of product states

Example 1

$$|0\rangle \otimes |1\rangle = |01\rangle .$$

Example 2: expand this tensor product.

$$|+\rangle \otimes |-\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle).$$

$$|+\rangle \otimes |-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle).$$

Check normalization: $4 \cdot \left|\frac{1}{2}\right|^2 = 1.$

General two-qubit states

A general two-qubit state is a normalized linear combination of the four basis states:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

General two-qubit states

A general two-qubit state is a normalized linear combination of the four basis states:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

For example,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad \frac{1}{\sqrt{6}}|01\rangle - \frac{1}{\sqrt{6}}|10\rangle + \sqrt{\frac{2}{3}}|11\rangle.$$

Must every two-qubit state be a product state?

More qubits, larger state spaces

The Hilbert space of n qubits is

$$\underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n}.$$

It has dimension 2^n and computational basis

$$\{|x\rangle : x \in \{0, 1\}^n\}.$$

More qubits, larger state spaces

The Hilbert space of n qubits is

$$\underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n}.$$

It has dimension 2^n and computational basis

$$\{|x\rangle : x \in \{0, 1\}^n\}.$$

A general state is

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle, \quad \sum_x |\alpha_x|^2 = 1.$$

Each additional qubit doubles the dimension.

Entangled States

Product states and entangled states

A pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a **product state** if

$$|\psi\rangle = |a\rangle \otimes |b\rangle$$

for some states $|a\rangle \in \mathcal{H}_A$ and $|b\rangle \in \mathcal{H}_B$.

Product states and entangled states

A pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a **product state** if

$$|\psi\rangle = |a\rangle \otimes |b\rangle$$

for some states $|a\rangle \in \mathcal{H}_A$ and $|b\rangle \in \mathcal{H}_B$.

It is **entangled** if no such factorization exists.

- To show a state is **not entangled**, exhibit a factorization.
- To show a state **is entangled**, prove that no factorization is possible.

Showing that a state is not entangled

Is this state entangled?

$$|\psi\rangle = \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle).$$

Showing that a state is not entangled

Is this state entangled?

$$|\psi\rangle = \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle).$$

Factor the expression:

$$\begin{aligned} |\psi\rangle &= \frac{1}{2} (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \\ &= |-\rangle \otimes |-\rangle. \end{aligned}$$

Showing that a state is not entangled

Is this state entangled?

$$|\psi\rangle = \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle).$$

Factor the expression:

$$\begin{aligned} |\psi\rangle &= \frac{1}{2} (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \\ &= |-\rangle \otimes |-\rangle. \end{aligned}$$

Not entangled. A superposition of several basis states can still be a product state.

Showing that the EPR pair is entangled

Consider the **EPR pair**, also called a **Bell pair**:

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle .$$

Showing that the EPR pair is entangled

Consider the **EPR pair**, also called a **Bell pair**:

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle .$$

Suppose, for a contradiction, that $|\text{EPR}\rangle = |a\rangle \otimes |b\rangle$, where

$$|a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle , \quad |b\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle .$$

Showing that the EPR pair is entangled

Consider the **EPR pair**, also called a **Bell pair**:

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle .$$

Suppose, for a contradiction, that $|\text{EPR}\rangle = |a\rangle \otimes |b\rangle$, where

$$|a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle , \quad |b\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle .$$

Then

$$|a\rangle \otimes |b\rangle = \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle .$$

Compare the four coefficients.

Showing that the EPR pair is entangled

Matching coefficients gives

$$\alpha_0\beta_0 = \frac{1}{\sqrt{2}}, \quad \alpha_0\beta_1 = 0, \quad \alpha_1\beta_0 = 0, \quad \alpha_1\beta_1 = \frac{1}{\sqrt{2}}.$$

Showing that the EPR pair is entangled

Matching coefficients gives

$$\alpha_0\beta_0 = \frac{1}{\sqrt{2}}, \quad \alpha_0\beta_1 = 0, \quad \alpha_1\beta_0 = 0, \quad \alpha_1\beta_1 = \frac{1}{\sqrt{2}}.$$

The first and last equations imply that

$$\alpha_0, \alpha_1, \beta_0, \beta_1 \text{ are all nonzero.}$$

Showing that the EPR pair is entangled

Matching coefficients gives

$$\alpha_0\beta_0 = \frac{1}{\sqrt{2}}, \quad \alpha_0\beta_1 = 0, \quad \alpha_1\beta_0 = 0, \quad \alpha_1\beta_1 = \frac{1}{\sqrt{2}}.$$

The first and last equations imply that

$$\alpha_0, \alpha_1, \beta_0, \beta_1 \text{ are all nonzero.}$$

But then $\alpha_0\beta_1$ and $\alpha_1\beta_0$ cannot be zero.

Contradiction. Therefore the EPR pair is entangled.

Unitary evolution of composite systems

If U acts on A and V acts on B , their joint action is $U \otimes V$:

$$(U \otimes V)(|a\rangle \otimes |b\rangle) = U|a\rangle \otimes V|b\rangle.$$

Unitary evolution of composite systems

If U acts on A and V acts on B , their joint action is $U \otimes V$:

$$(U \otimes V)(|a\rangle \otimes |b\rangle) = U|a\rangle \otimes V|b\rangle.$$

For a general state, extend this rule by linearity:

$$(U \otimes V) \sum_{ij} \alpha_{ij} |ij\rangle = \sum_{ij} \alpha_{ij} U|i\rangle \otimes V|j\rangle.$$

Unitary evolution of composite systems

If U acts on A and V acts on B , their joint action is $U \otimes V$:

$$(U \otimes V)(|a\rangle \otimes |b\rangle) = U|a\rangle \otimes V|b\rangle.$$

For a general state, extend this rule by linearity:

$$(U \otimes V) \sum_{ij} \alpha_{ij} |ij\rangle = \sum_{ij} \alpha_{ij} U|i\rangle \otimes V|j\rangle.$$

Example

$$(H \otimes I) |00\rangle = |+\rangle \otimes |0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

Not every two-qubit unitary has the form $U \otimes V$.

CNOT creates entanglement

CNOT flips the second qubit when the first qubit is $|1\rangle$:

$$\text{CNOT } |x, y\rangle = |x, y \oplus x\rangle.$$



CNOT creates entanglement

CNOT flips the second qubit when the first qubit is $|1\rangle$:

$$\text{CNOT } |x, y\rangle = |x, y \oplus x\rangle.$$



$$|00\rangle \xrightarrow{H \otimes I} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

A product state becomes an entangled state.

Partial Measurements

Measuring both qubits

Suppose

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |ij\rangle .$$

Measuring **both qubits in the computational basis** gives

$$\Pr[\text{outcome } (i,j)] = |\alpha_{ij}|^2, \quad \text{post-measurement state: } |ij\rangle .$$

Measuring both qubits

Suppose

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |ij\rangle .$$

Measuring **both qubits in the computational basis** gives

$$\Pr[\text{outcome } (i,j)] = |\alpha_{ij}|^2, \quad \text{post-measurement state: } |ij\rangle .$$

Example: the EPR pair

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \longrightarrow \begin{cases} |00\rangle & \text{with probability } \frac{1}{2}, \\ |11\rangle & \text{with probability } \frac{1}{2}. \end{cases}$$

Measuring both qubits: examples

$$|\psi_1\rangle = |-\rangle \otimes |-\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{6}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle.$$

What are the outcome distributions?

Measuring both qubits: examples

$$|\psi_1\rangle = |-\rangle \otimes |-\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{6}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle.$$

What are the outcome distributions?

	00	01	10	11
Measure $ \psi_1\rangle$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Measure $ \psi_2\rangle$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$

In either case, the joint state becomes the basis state observed.

What happens if we measure **only the second qubit**, B ?

1. What is the distribution of outcomes?
2. What is the state of the system **conditioned on the outcome**?

Partial measurements

What happens if we measure **only the second qubit**, B ?

1. What is the distribution of outcomes?
2. What is the state of the system **conditioned on the outcome**?

We need to keep track of both the **classical outcome** and the **remaining quantum state**.

Partial measurement of a product state

Measure B in

$$|\psi\rangle = |-\rangle_A \otimes |-\rangle_B = \frac{1}{\sqrt{2}} |-\rangle_A \otimes |0\rangle_B - \frac{1}{\sqrt{2}} |-\rangle_A \otimes |1\rangle_B.$$

Partial measurement of a product state

Measure B in

$$|\psi\rangle = |-\rangle_A \otimes |-\rangle_B = \frac{1}{\sqrt{2}} |-\rangle_A \otimes |0\rangle_B - \frac{1}{\sqrt{2}} |-\rangle_A \otimes |1\rangle_B.$$

Outcome on B	Probability	Joint state afterwards
0	$\frac{1}{2}$	$ -\rangle_A \otimes 0\rangle_B$
1	$\frac{1}{2}$	$ -\rangle_A \otimes 1\rangle_B$

Post-measurement states are written up to global phase.

Partial measurement of a product state

Measure B in

$$|\psi\rangle = |-\rangle_A \otimes |-\rangle_B = \frac{1}{\sqrt{2}} |-\rangle_A \otimes |0\rangle_B - \frac{1}{\sqrt{2}} |-\rangle_A \otimes |1\rangle_B.$$

Outcome on B	Probability	Joint state afterwards
0	$\frac{1}{2}$	$ -\rangle_A \otimes 0\rangle_B$
1	$\frac{1}{2}$	$ -\rangle_A \otimes 1\rangle_B$

Post-measurement states are written up to global phase.

The unmeasured qubit remains in state $|-\rangle$.

Partial measurement of an entangled state

Measure B in

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle .$$

Partial measurement of an entangled state

Measure B in

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle.$$

Outcome on B	Probability	Joint state	State of A
0	$\frac{1}{2}$	$ 00\rangle$	$ 0\rangle$
1	$\frac{1}{2}$	$ 11\rangle$	$ 1\rangle$

Partial measurement of an entangled state

Measure B in

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle.$$

Outcome on B	Probability	Joint state	State of A
0	$\frac{1}{2}$	$ 00\rangle$	$ 0\rangle$
1	$\frac{1}{2}$	$ 11\rangle$	$ 1\rangle$

The conditional state of A depends on the outcome on B .

A nonuniform example: outcome probabilities

Measure the **second qubit** of

$$|\psi\rangle = \frac{1}{\sqrt{6}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle.$$

Group terms by the value of the second bit:

$$|\psi\rangle = \underbrace{\frac{1}{\sqrt{6}} |0\rangle}_{\text{second bit 0}} \otimes |0\rangle + \underbrace{\left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle \right)}_{\text{second bit 1}} \otimes |1\rangle.$$

A nonuniform example: outcome probabilities

Measure the **second qubit** of

$$|\psi\rangle = \frac{1}{\sqrt{6}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle.$$

Group terms by the value of the second bit:

$$|\psi\rangle = \underbrace{\frac{1}{\sqrt{6}} |0\rangle}_{\text{second bit 0}} \otimes |0\rangle + \underbrace{\left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle \right)}_{\text{second bit 1}} \otimes |1\rangle.$$

$$\Pr[B = 0] = \frac{1}{6}, \quad \Pr[B = 1] = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Add the **squared magnitudes** of the amplitudes consistent with each outcome.

A nonuniform example: conditional states

Outcome 0 has probability $p_0 = \frac{1}{6}$. The surviving branch, divided by $\sqrt{p_0}$, is

$$\frac{\frac{1}{\sqrt{6}} |00\rangle}{\sqrt{1/6}} = |00\rangle .$$

A nonuniform example: conditional states

Outcome 0 has probability $p_0 = \frac{1}{6}$. The surviving branch, divided by $\sqrt{p_0}$, is

$$\frac{\frac{1}{\sqrt{6}} |00\rangle}{\sqrt{1/6}} = |00\rangle.$$

Outcome 1 has probability $p_1 = \frac{5}{6}$. The surviving branch, divided by $\sqrt{p_1}$, is

$$\begin{aligned} \frac{\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle}{\sqrt{5/6}} &= \sqrt{\frac{3}{5}} |01\rangle + \sqrt{\frac{2}{5}} |11\rangle \\ &= \left(\sqrt{\frac{3}{5}} |0\rangle + \sqrt{\frac{2}{5}} |1\rangle \right) \otimes |1\rangle. \end{aligned}$$

A nonuniform example: conditional states

Outcome 0 has probability $p_0 = \frac{1}{6}$. The surviving branch, divided by $\sqrt{p_0}$, is

$$\frac{\frac{1}{\sqrt{6}} |00\rangle}{\sqrt{1/6}} = |00\rangle.$$

Outcome 1 has probability $p_1 = \frac{5}{6}$. The surviving branch, divided by $\sqrt{p_1}$, is

$$\begin{aligned} \frac{\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{3}} |11\rangle}{\sqrt{5/6}} &= \sqrt{\frac{3}{5}} |01\rangle + \sqrt{\frac{2}{5}} |11\rangle \\ &= \left(\sqrt{\frac{3}{5}} |0\rangle + \sqrt{\frac{2}{5}} |1\rangle \right) \otimes |1\rangle. \end{aligned}$$

Keep the consistent terms, then renormalize.

Partial measurements: the rule

Suppose

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B.$$

Measuring B in the computational basis gives

$$\Pr[B = j] = p_j = \sum_i |\alpha_{ij}|^2.$$

Partial measurements: the rule

Suppose

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B.$$

Measuring B in the computational basis gives

$$\Pr[B = j] = p_j = \sum_i |\alpha_{ij}|^2.$$

For an outcome j with $p_j > 0$, the joint state becomes

$$\frac{1}{\sqrt{p_j}} \sum_i \alpha_{ij} |i\rangle_A \otimes |j\rangle_B.$$

Keep the consistent terms, then renormalize.

Summary

1. **Composite systems:** combine spaces using tensor products; dimensions multiply.
2. **Entanglement:** a joint pure state need not factor into states of its subsystems.
3. **Partial measurements:** find the outcome probability, keep the consistent terms, and renormalize.