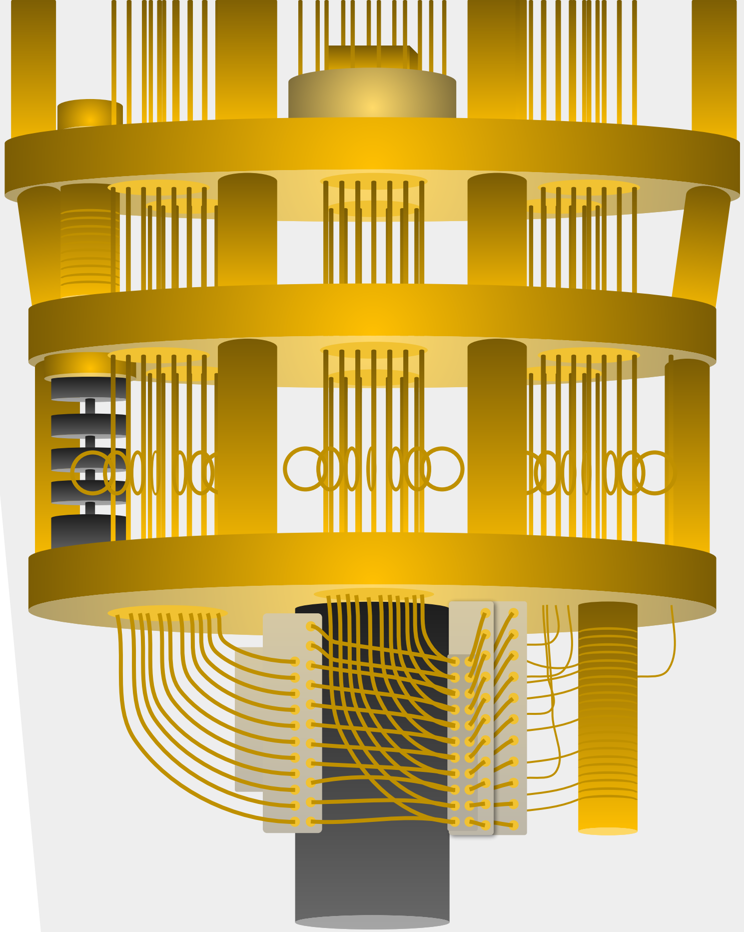
# Quantum Computing Basics (continued)



Barak Nehoran
Postdoc, CS Department, Columbia University

COMS 4281:
Intro to Quantum Computing
Sep 25, 2026

## Classical Information

## Quantum Information

Barak Nehoran (Postdoc, CS Department, Columbia University)

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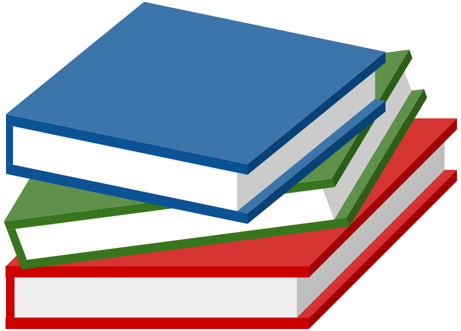

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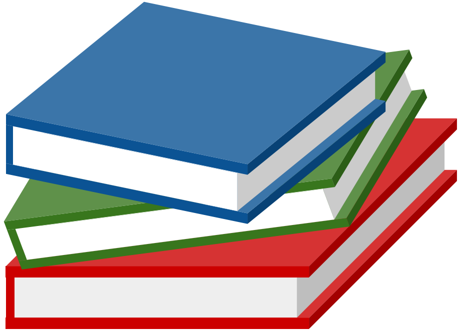

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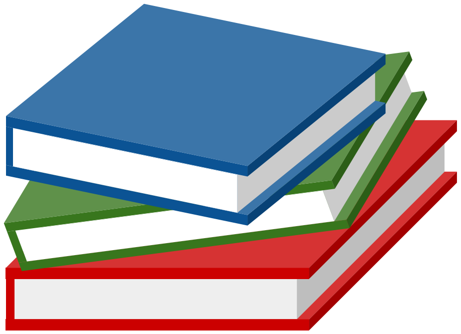

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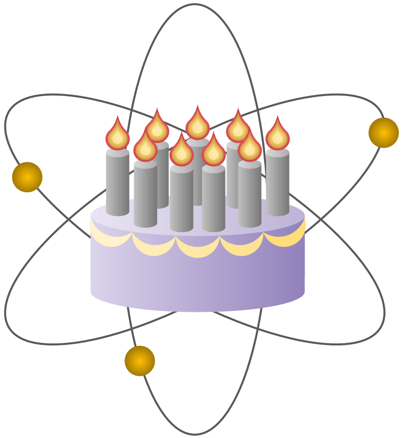

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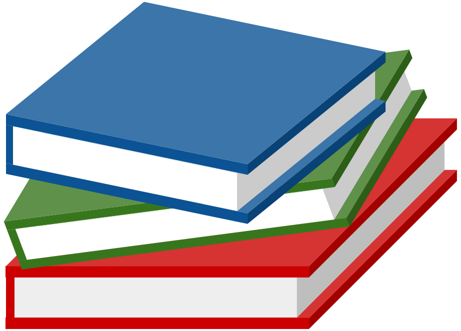

You can

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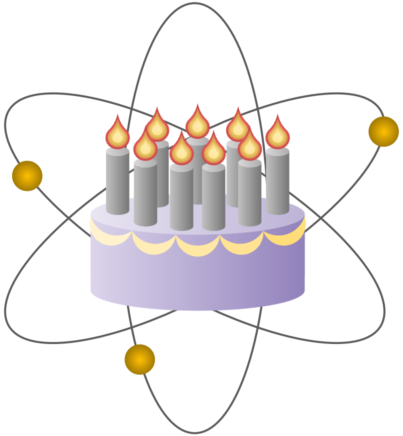

By contrast,

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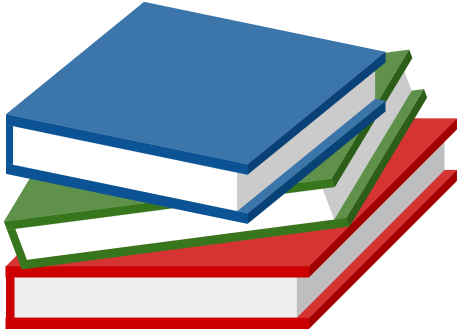

You can

1. read it 

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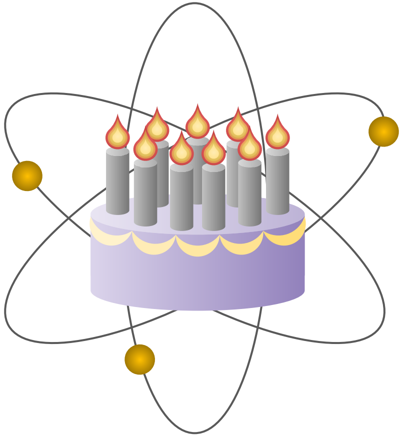

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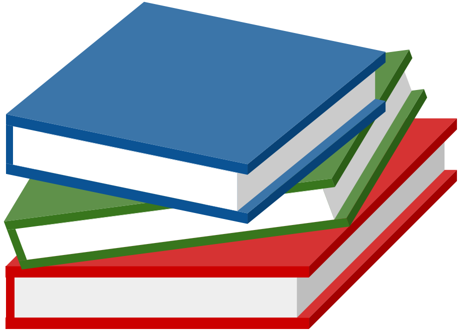

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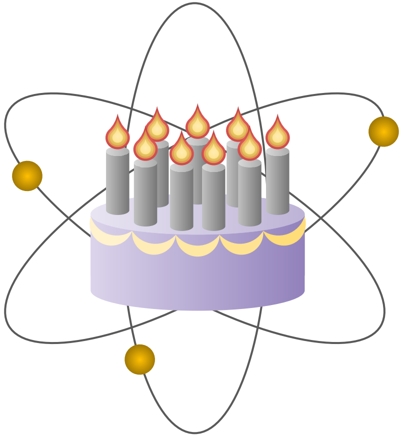

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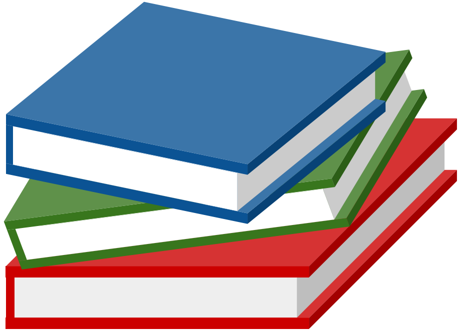

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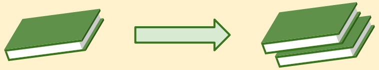

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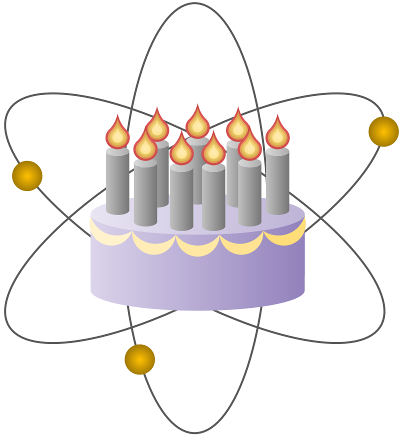

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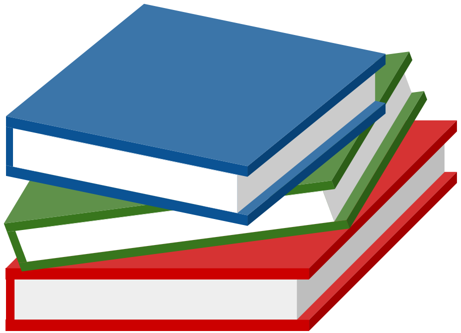

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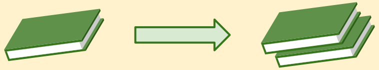

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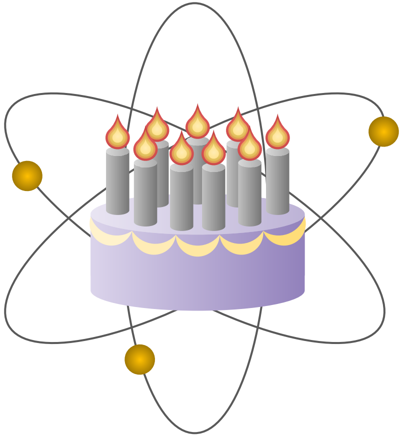

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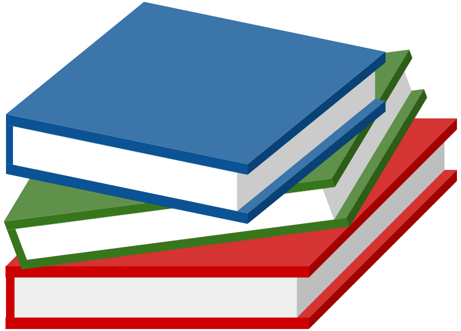

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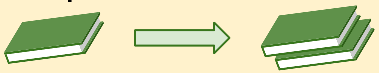
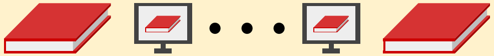


3. send it over the internet



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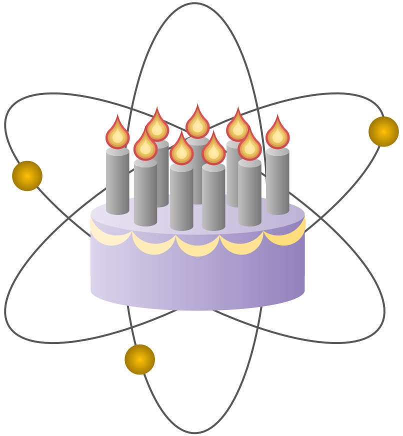

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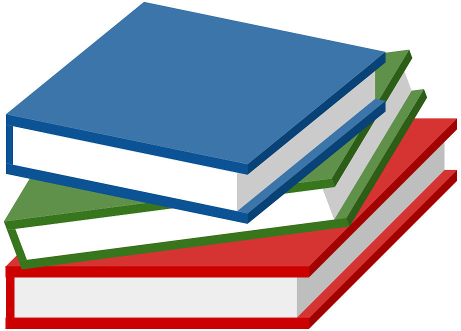

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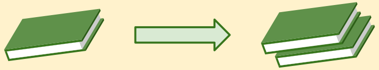

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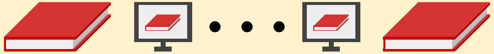

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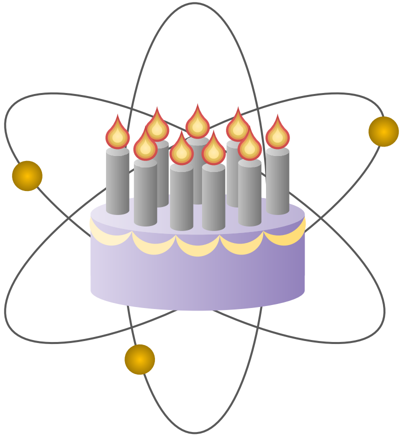

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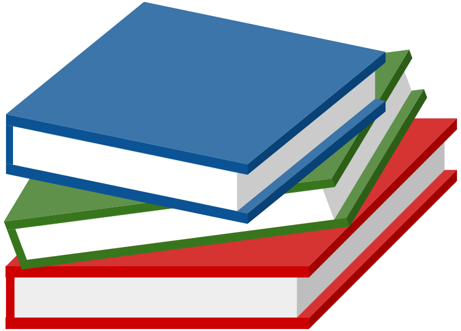

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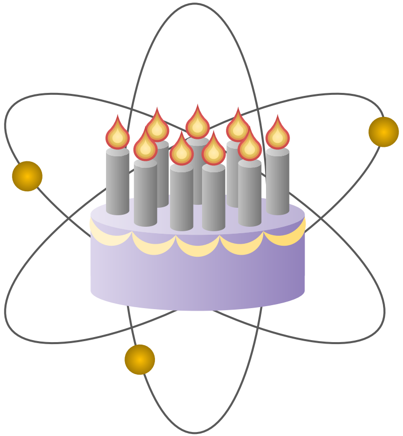

Measurement

Cloning

You can

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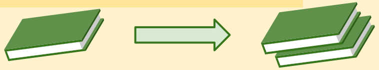
2. make copies of it



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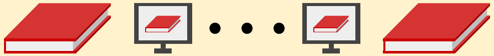

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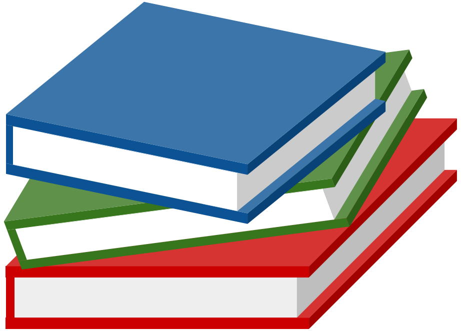

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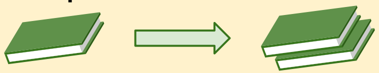

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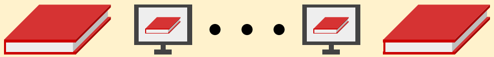

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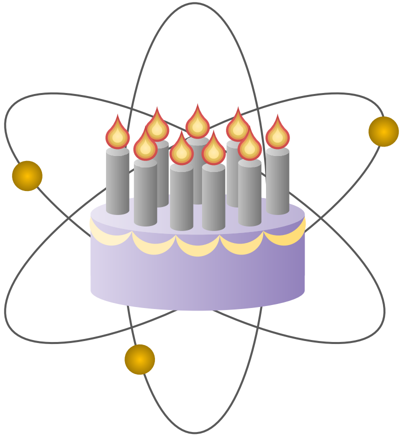

Barak Nehoran (Postdoc, CS Department, Columbia University)

# Outline

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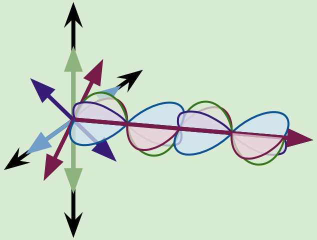
1 The Qubit



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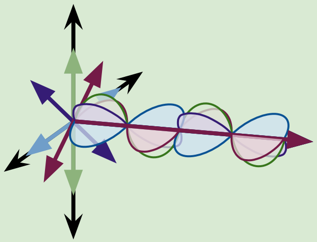

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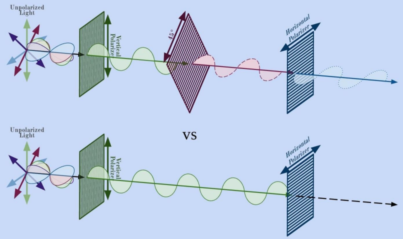

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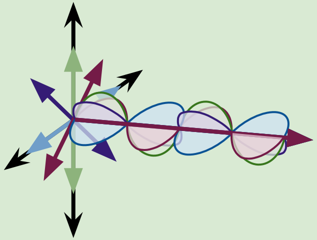

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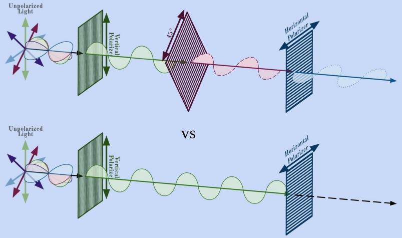


3 **Quantum Entanglement & Many Qubits**

First qubit  Second qubit

$$\frac{1}{\sqrt{2}}|0\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|0\rangle$$

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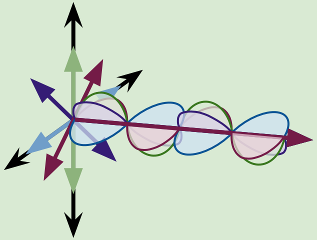

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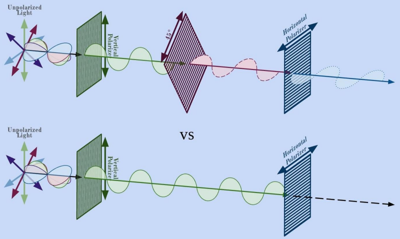


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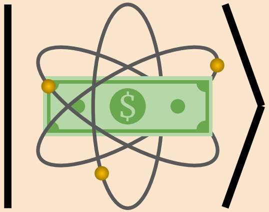
4 No Cloning and Quantum Money



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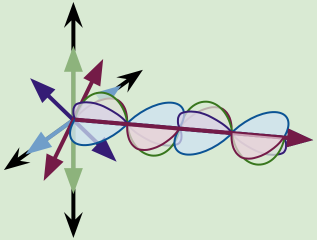

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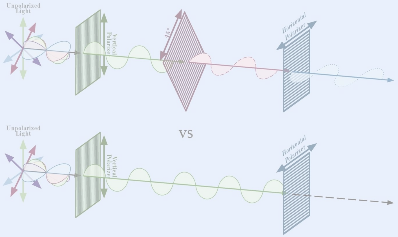


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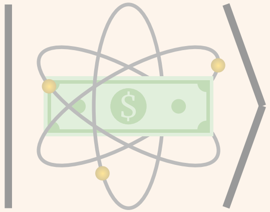

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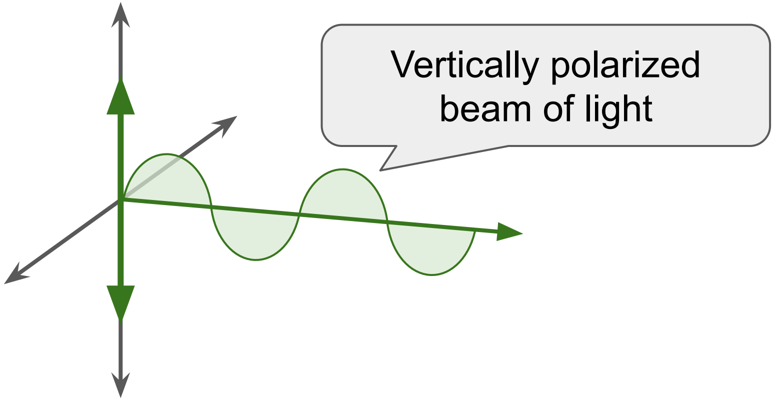
# Quantum States and the Qubit: *Using light polarization as a running intuition*



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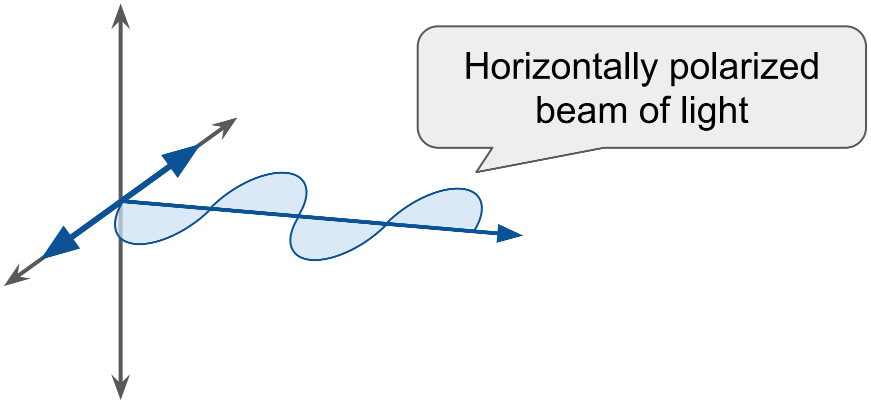

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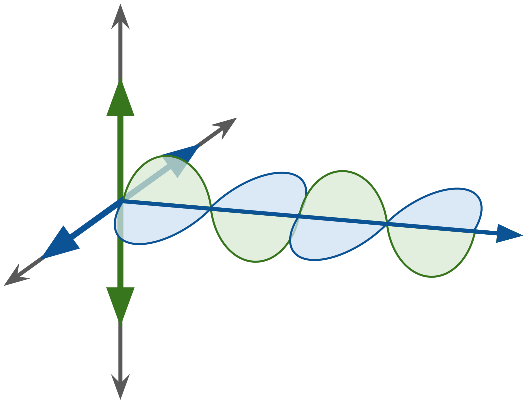

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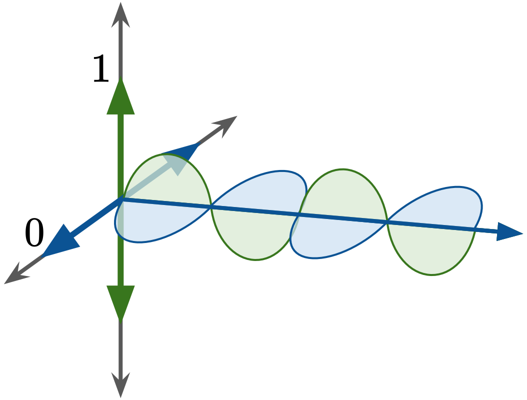


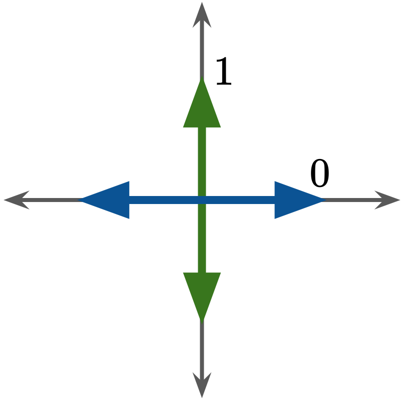


Representation of a **bit**

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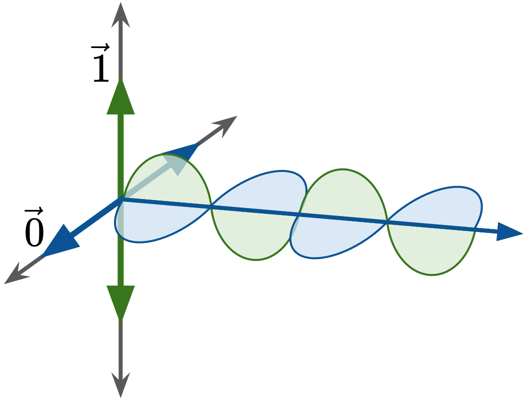


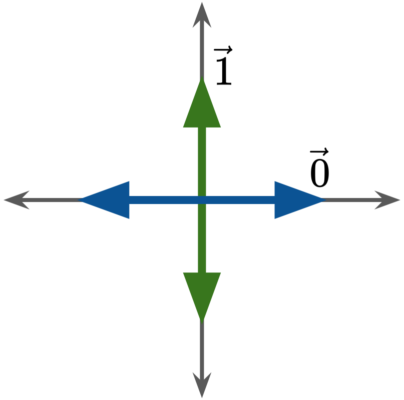


Representation of a *quantum* bit or "***qubit***"!

$$\vec{0} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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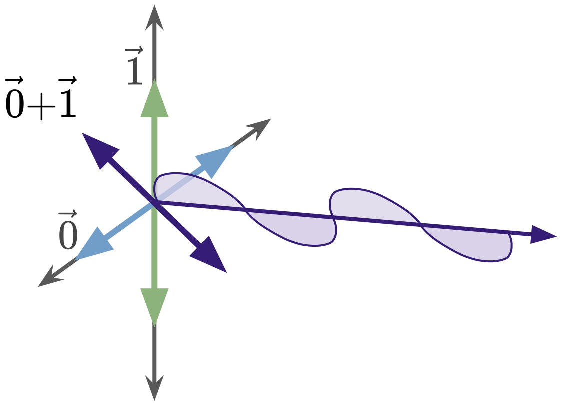


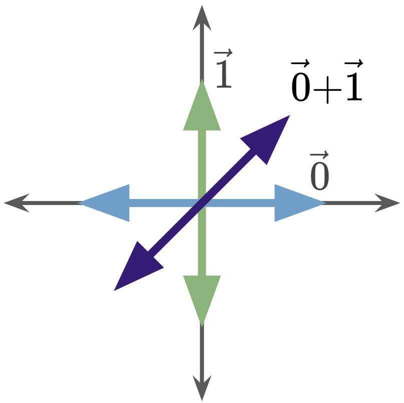

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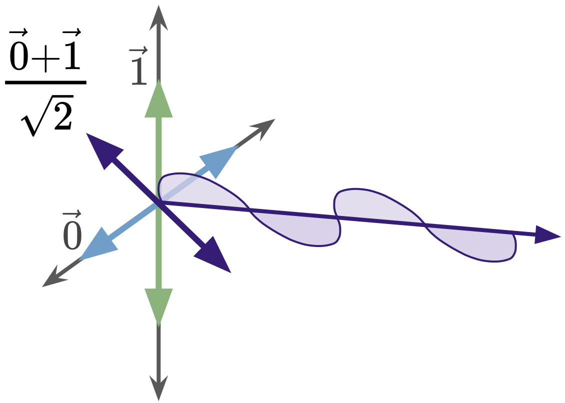


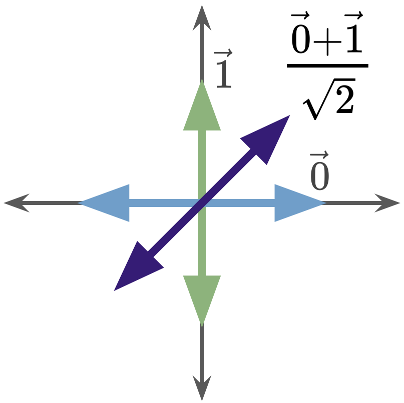

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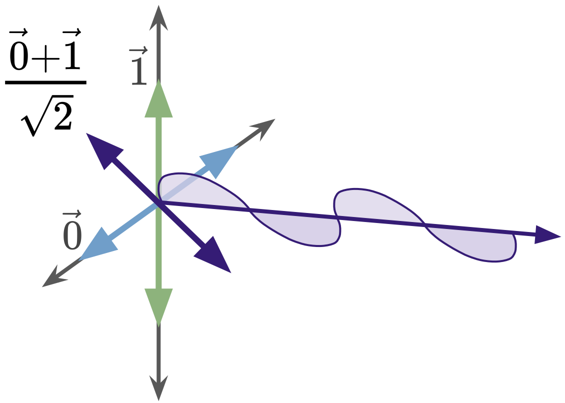


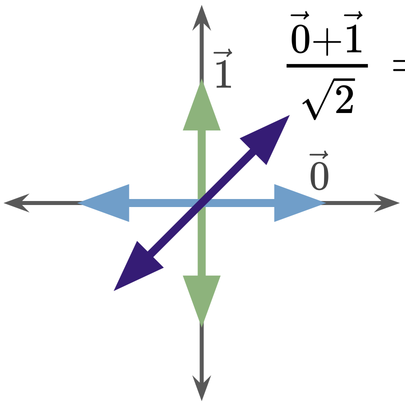


$$\frac{\vec{0}+\vec{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

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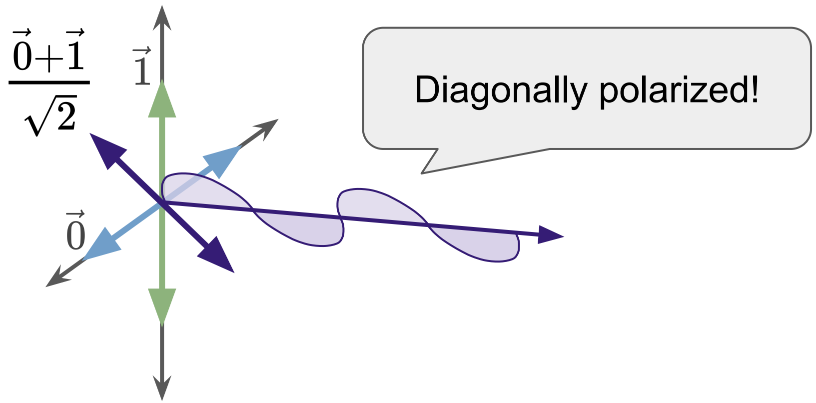


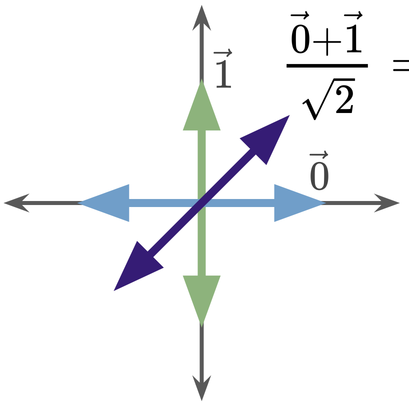


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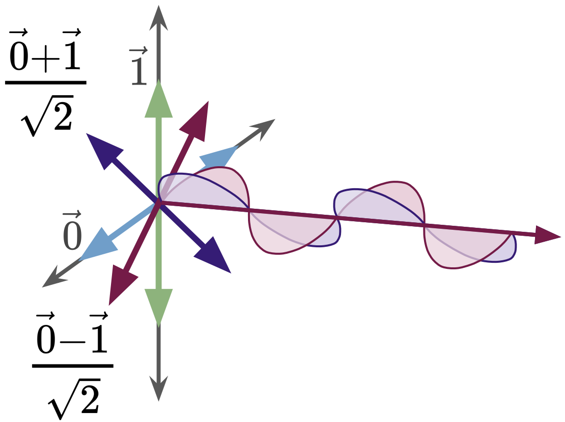
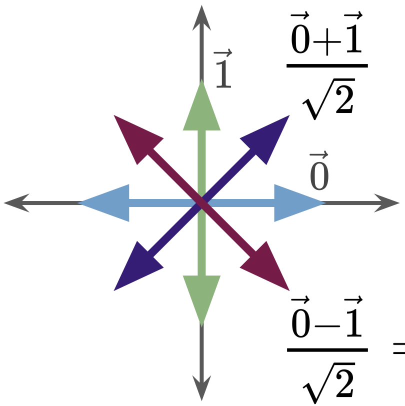
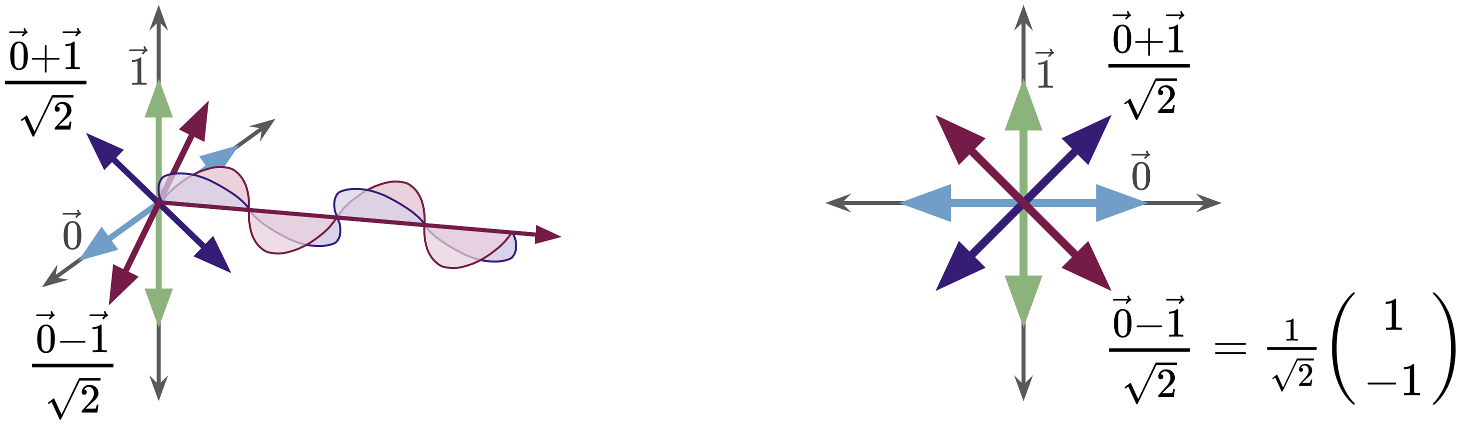



$$\frac{\vec{0}-\vec{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

# Quantum States and the Qubit: *Using light polarization as a running intuition*




You may have heard this described as

"***being simultaneously in both the zero and the one state***."

These linear combinations are really just another unit vector in a different direction!

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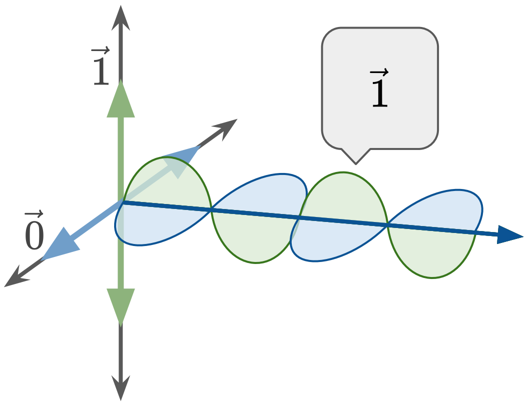
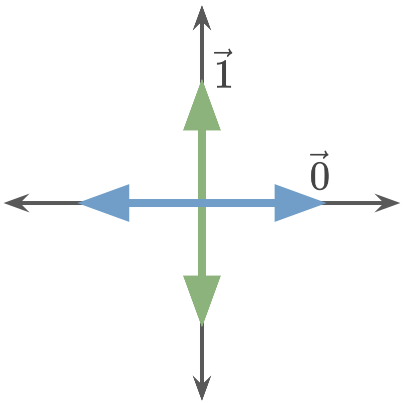
# Quantum States and the Qubit: *Using light polarization as a running intuition*



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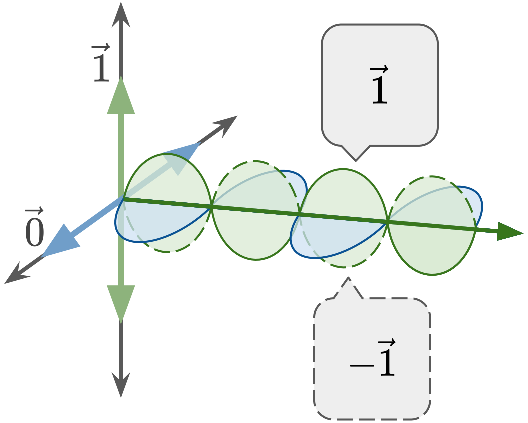


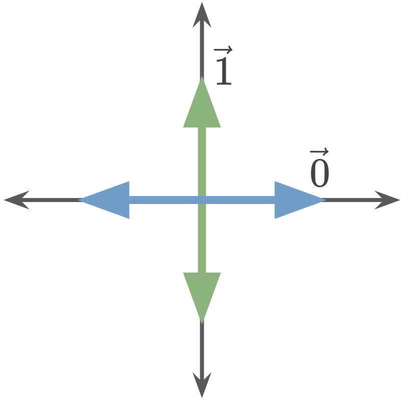


Observe that **negating a state** vector leaves it **polarized the same way**.

It merely **changes the phase!** $-\vec{v} \equiv \vec{v}$

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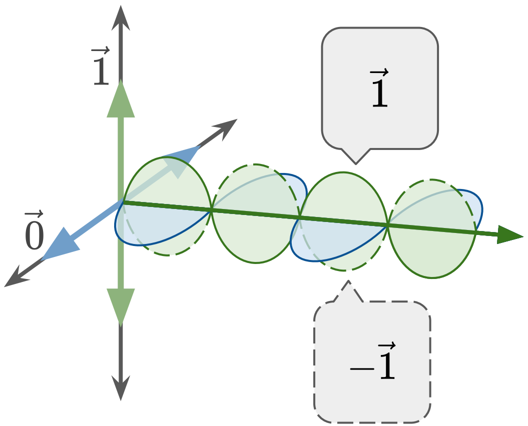
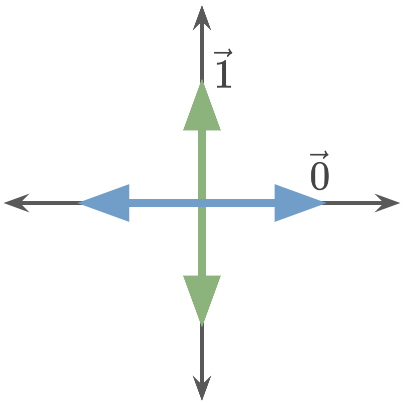


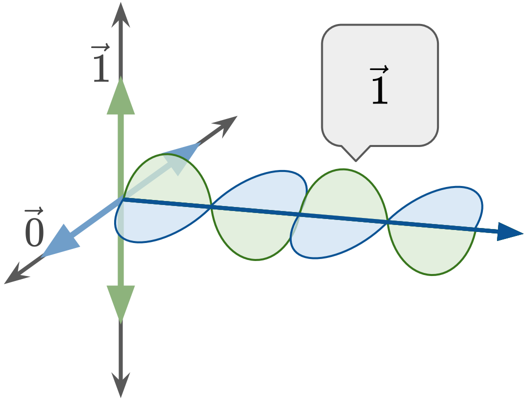

Observe that **negating a state** vector leaves it **polarized the same way**.

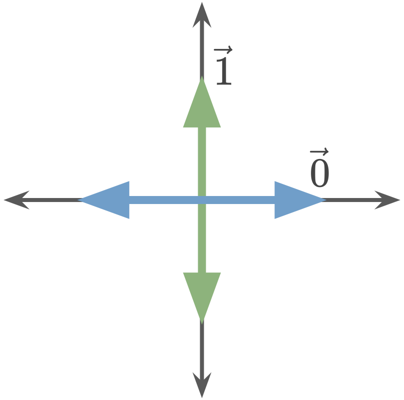
It merely **changes the phase**! $-\vec{v} \equiv \vec{v}$

In fact, that's the case in general for *any complex coefficient of unit norm*.

$$e^{i\theta}\vec{v} \equiv \vec{v} \ \ \forall\theta$$

# Aside: Complex Number Coefficients



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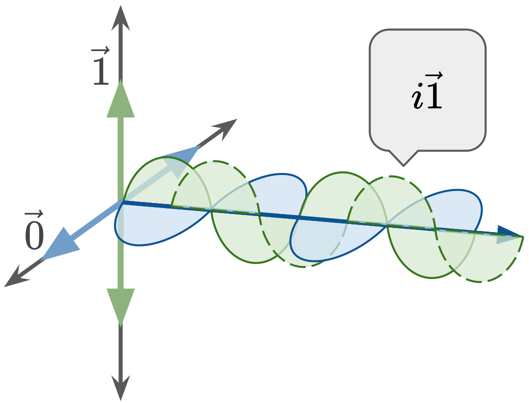


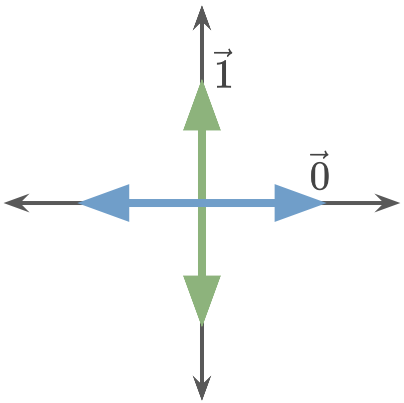

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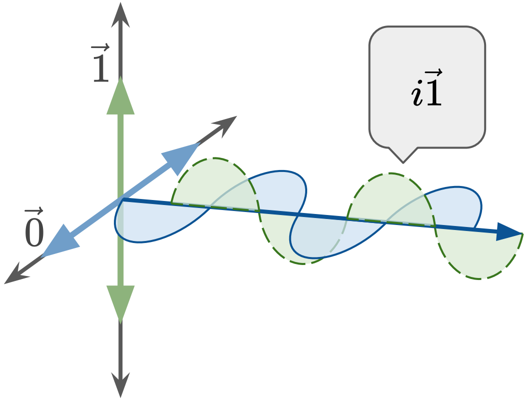


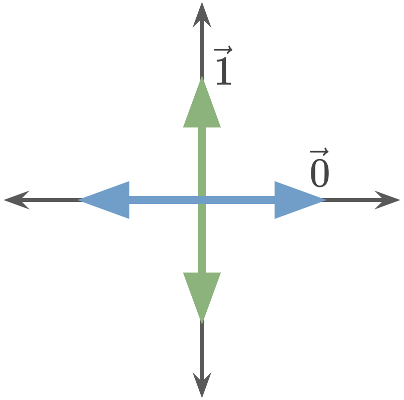

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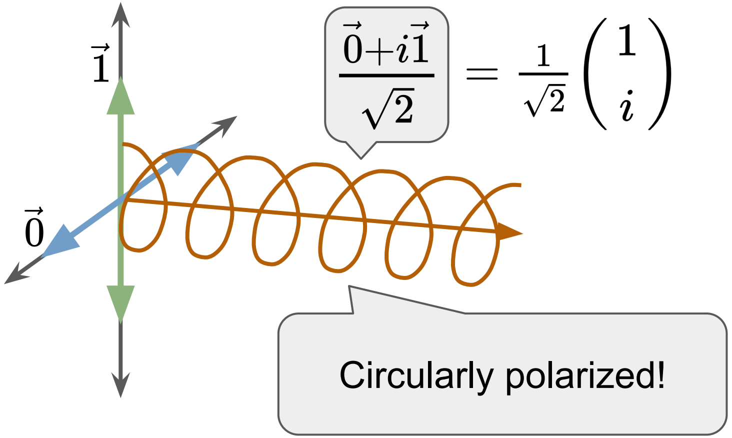


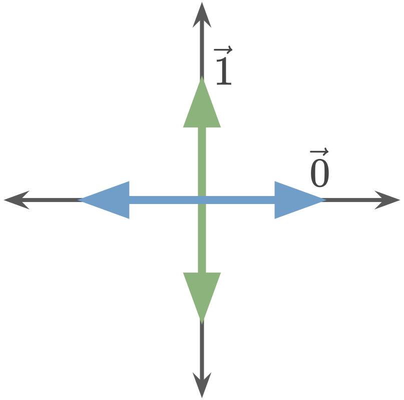

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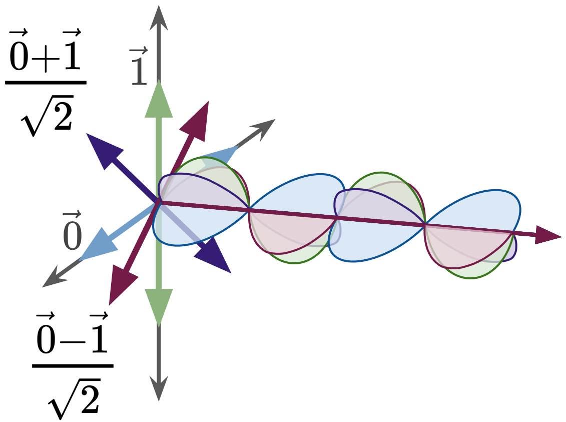


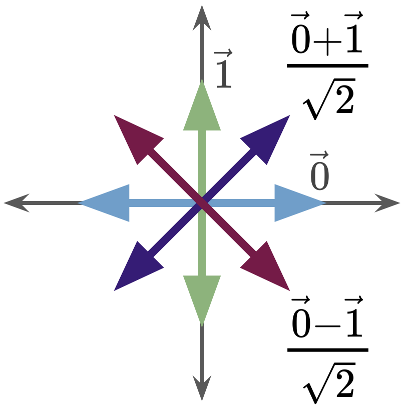

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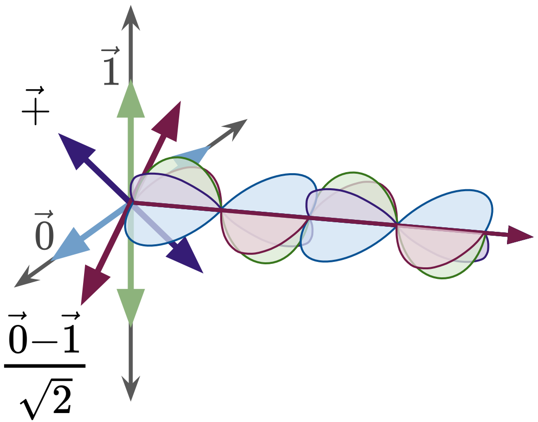


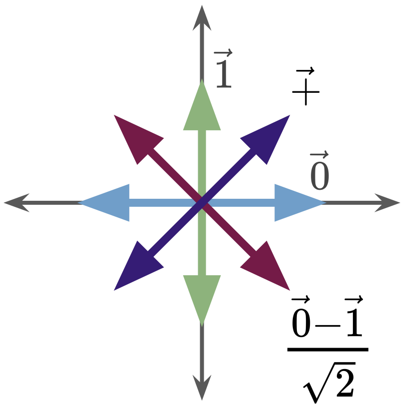

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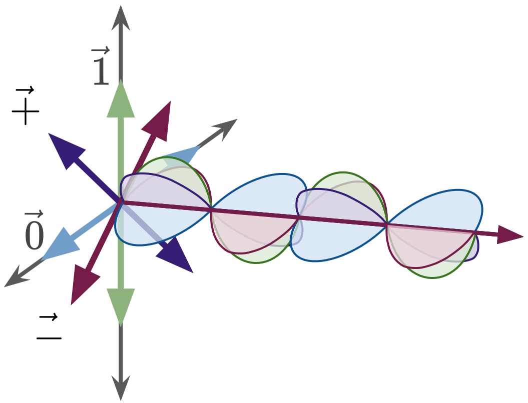


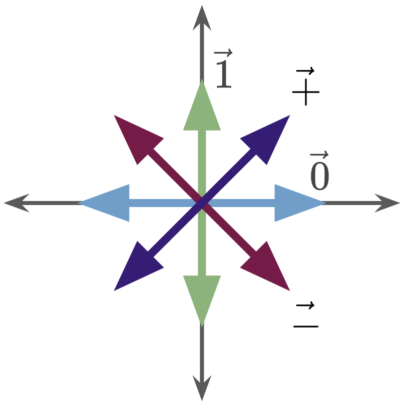

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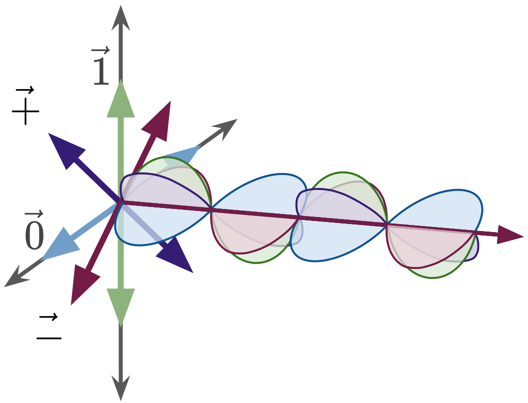
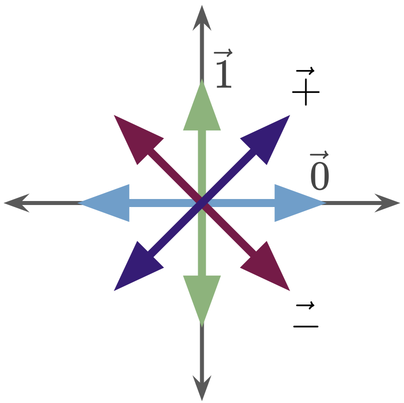


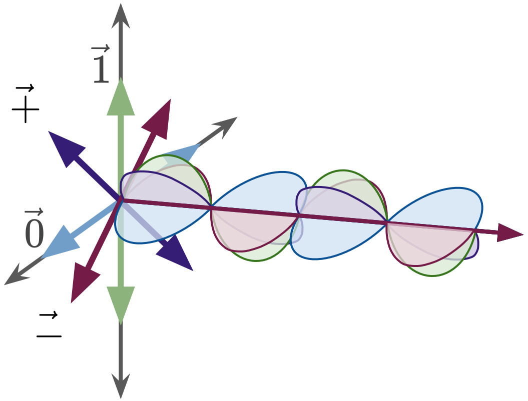




## Takeaways:

- A quantum state is represented by a **unit vector**.
- Any normalized linear combinations of quantum states is also a valid quantum state. This is called **superposition**.

# Examples of Qubits: *Qubits are everywhere!*

Light Polarization




The **qubit** is the basic unit of **quantum information**.

It is a quantum state on **any two-level system**.

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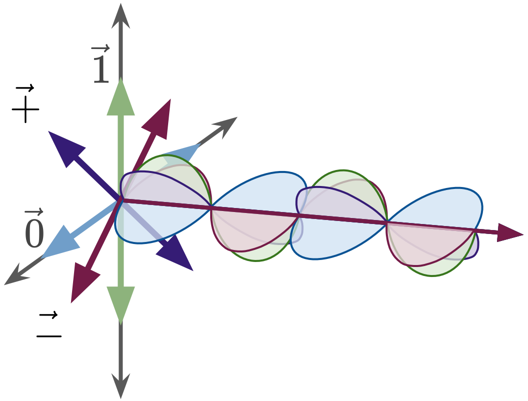


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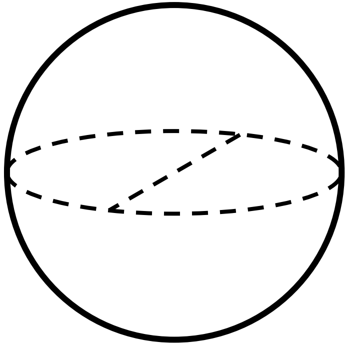

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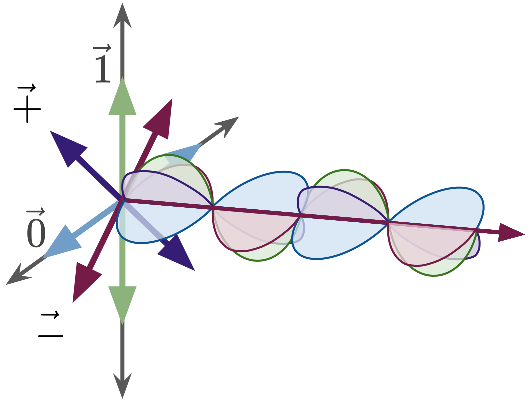


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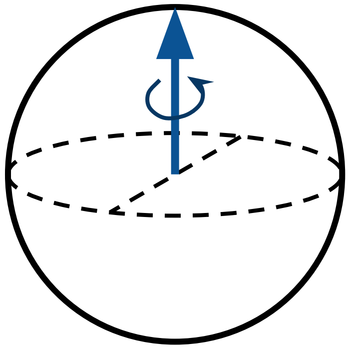

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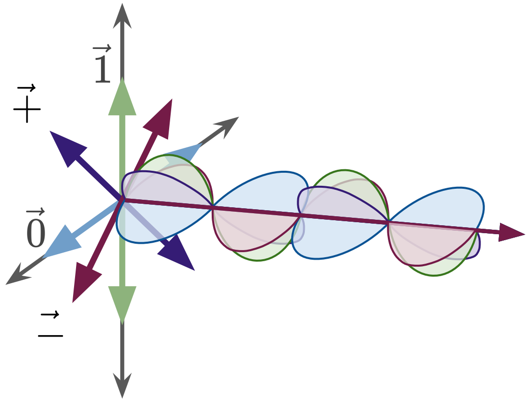


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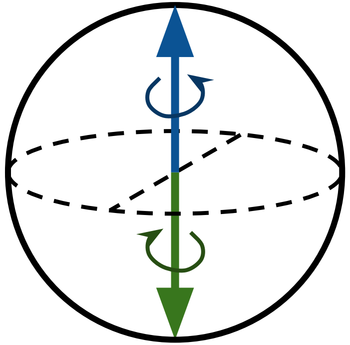

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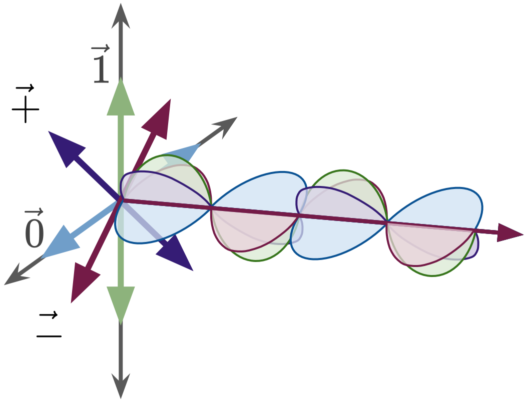


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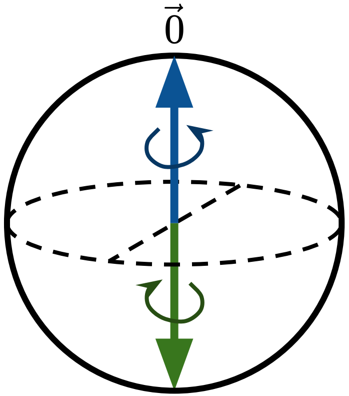


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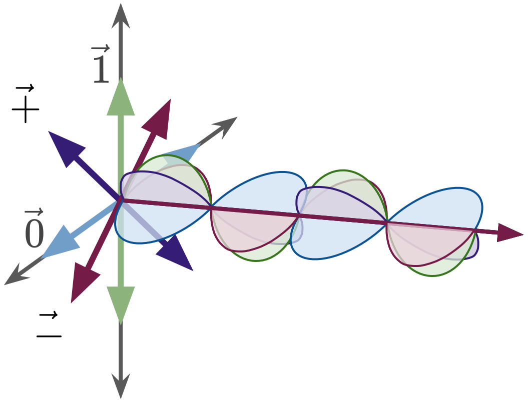


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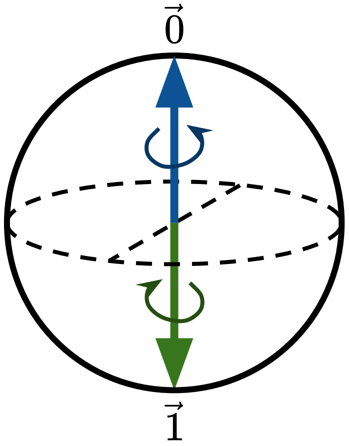


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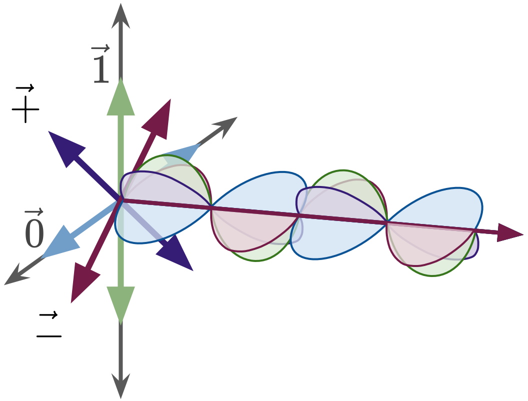


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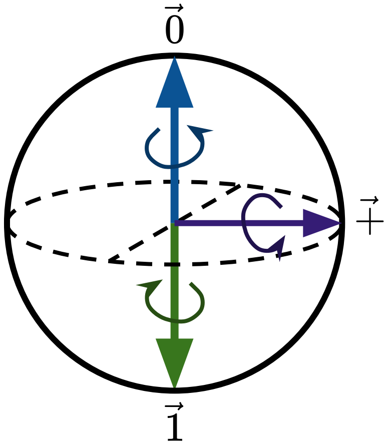


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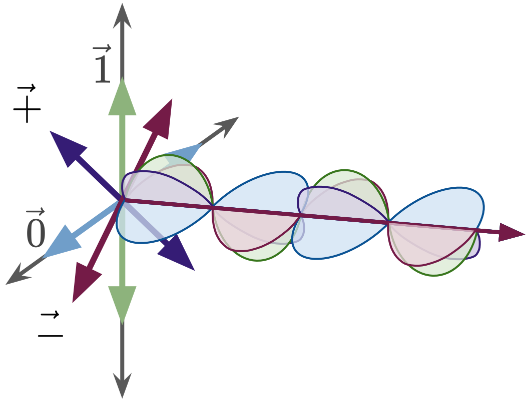


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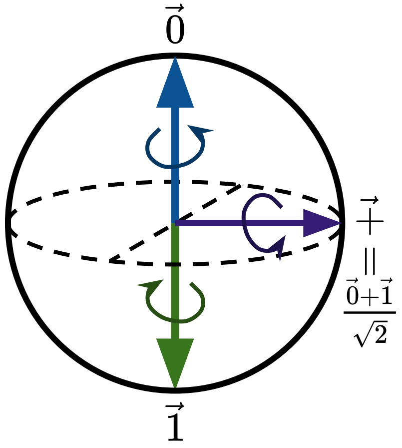


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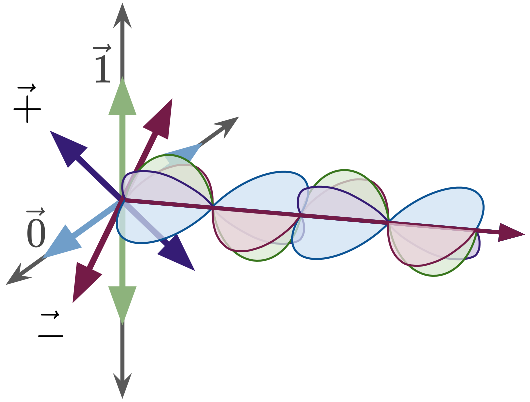


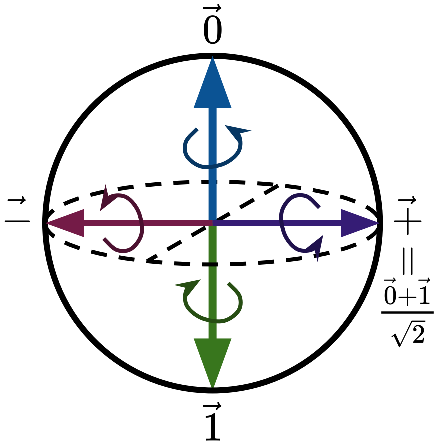


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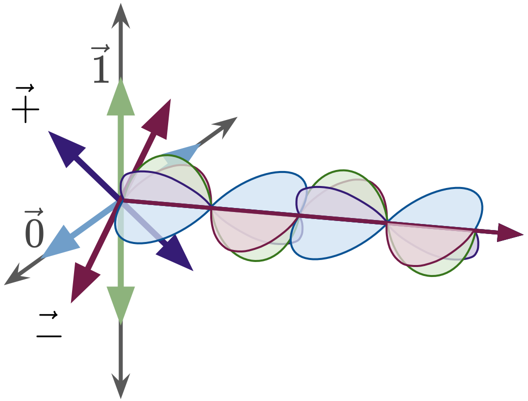


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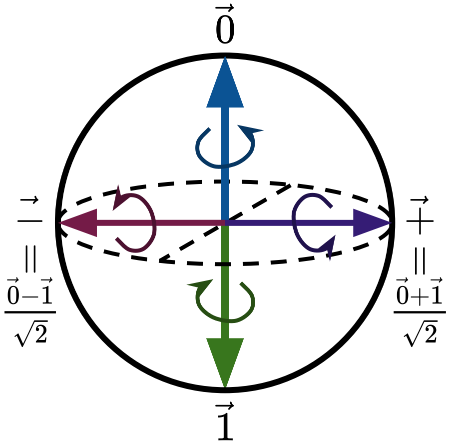


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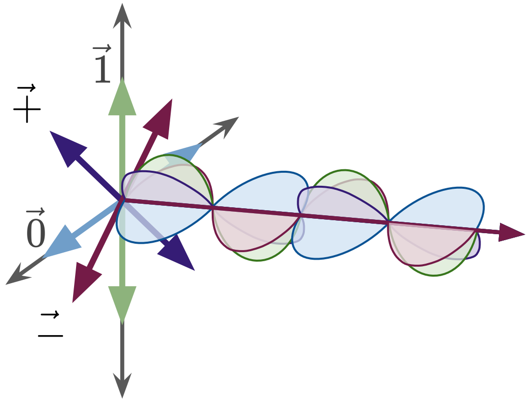


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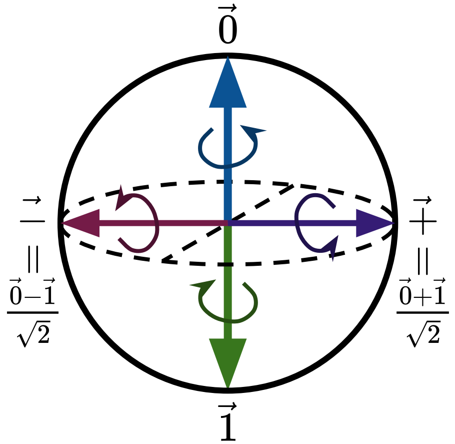
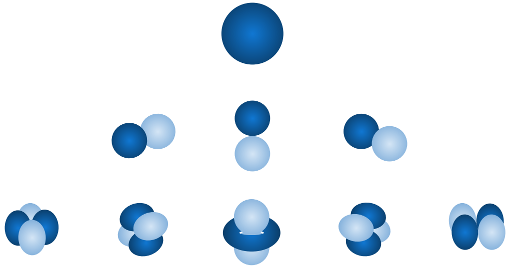



Electron Orbitals



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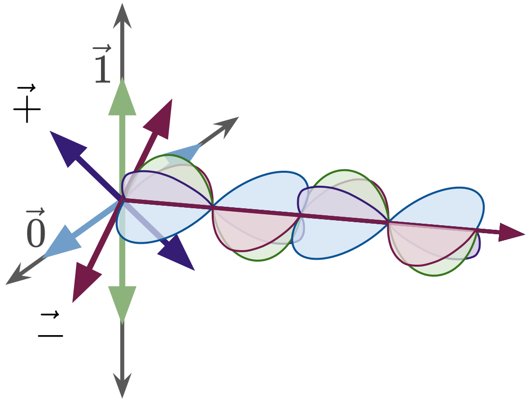


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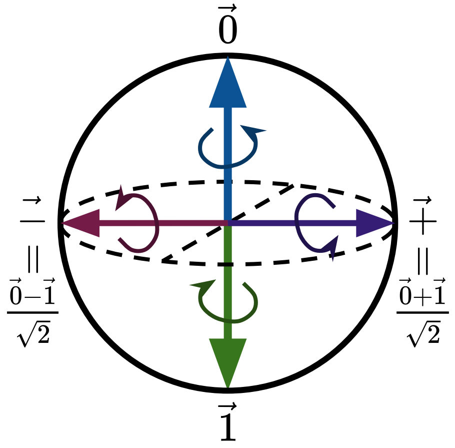


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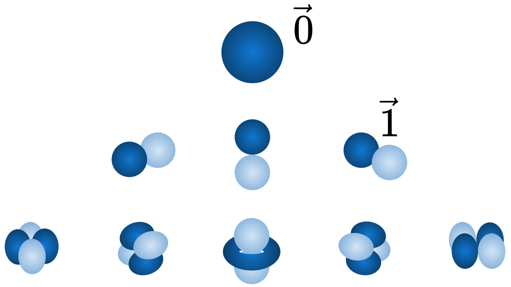


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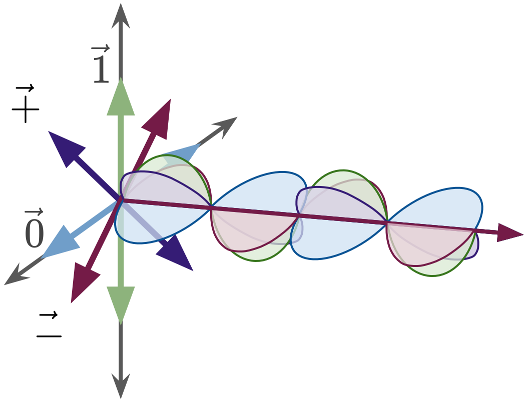


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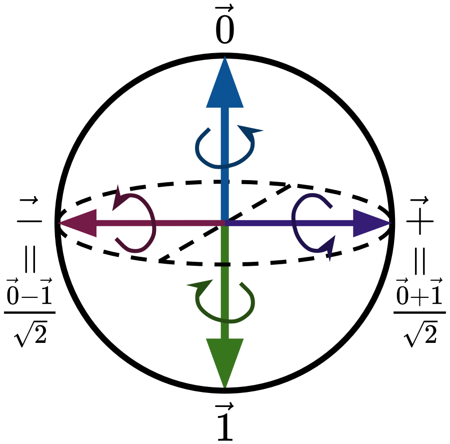


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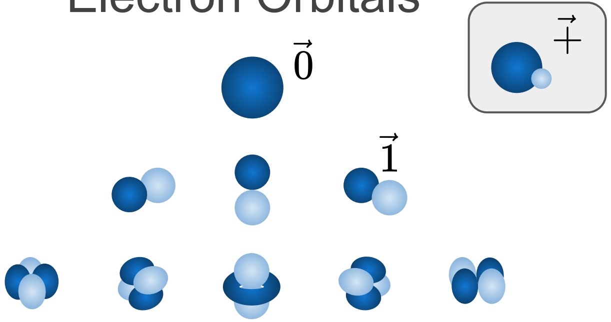


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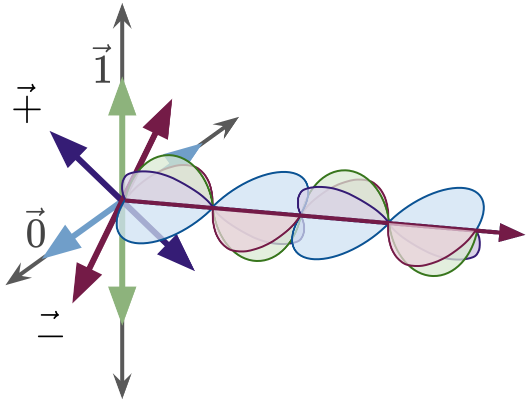


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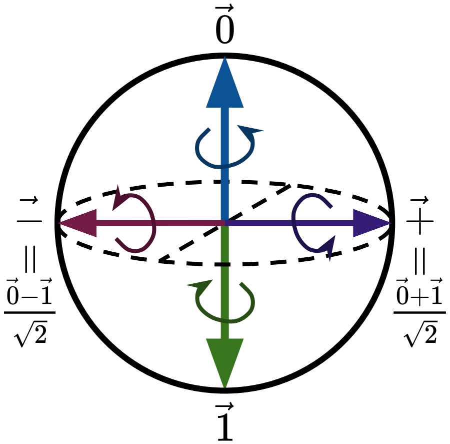


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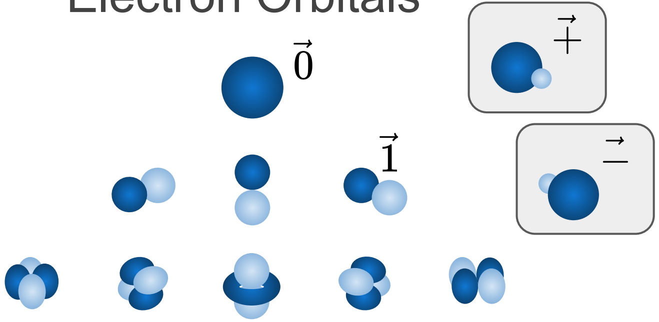


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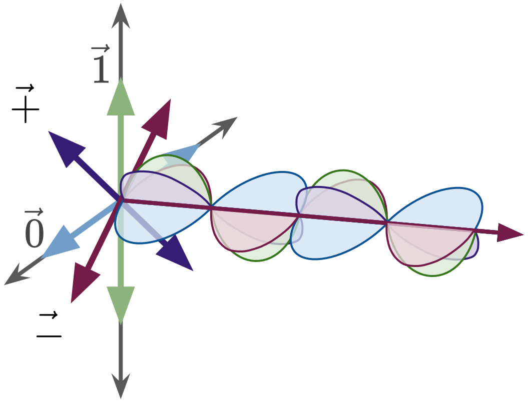


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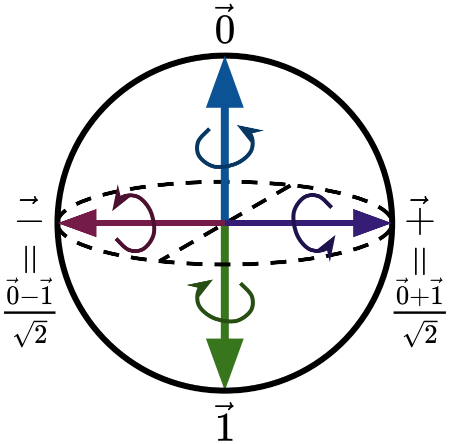


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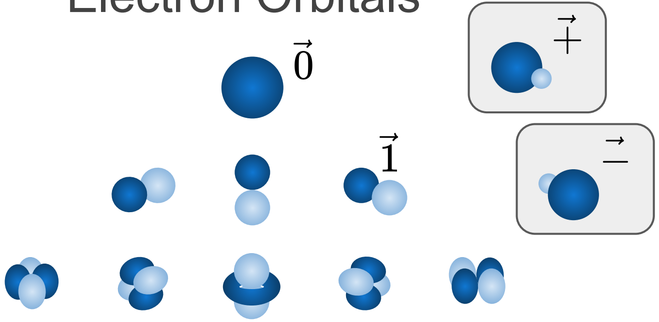
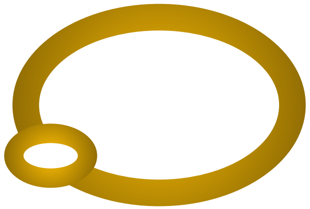



Superconducting Currents



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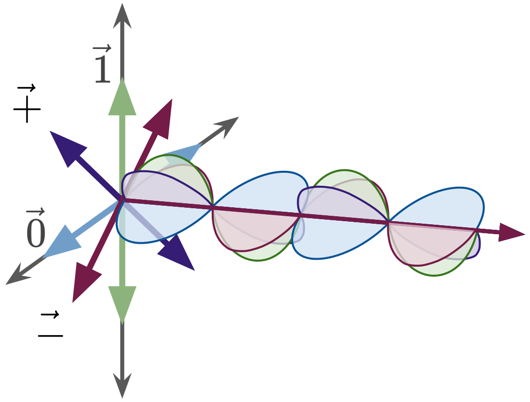


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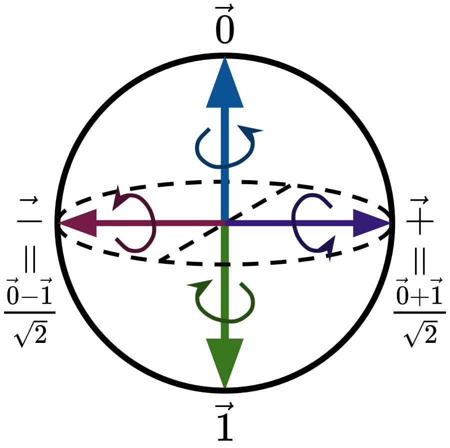


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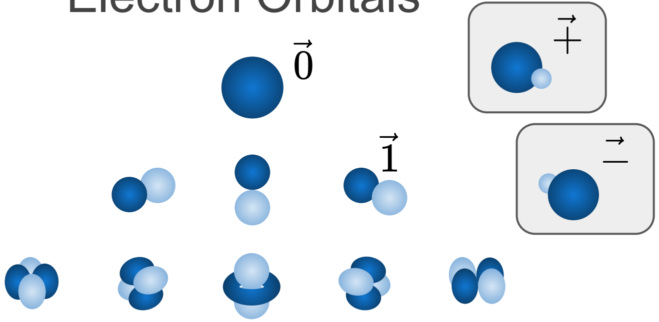


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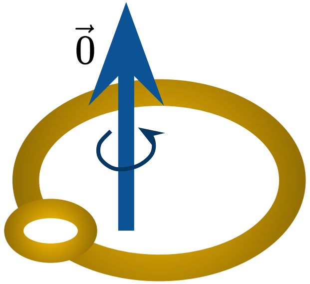


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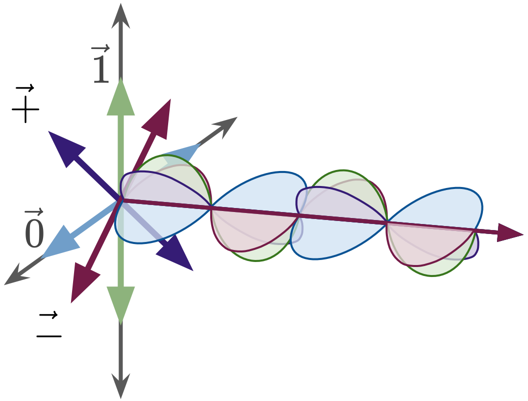


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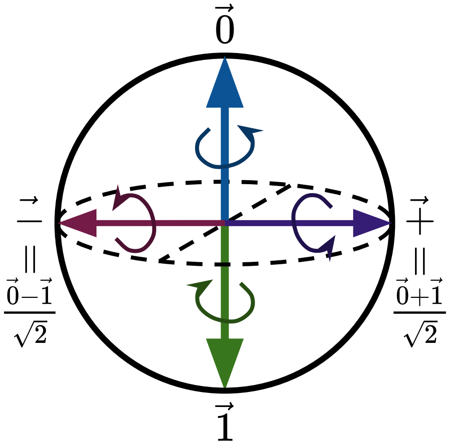


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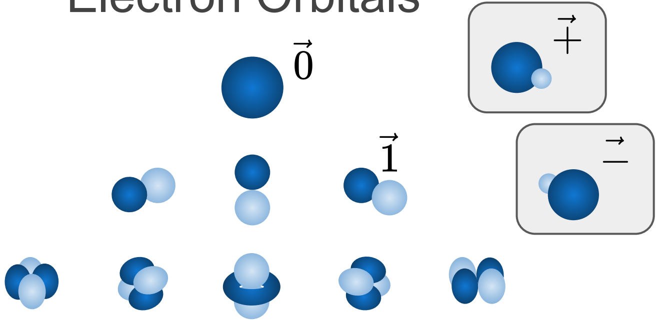


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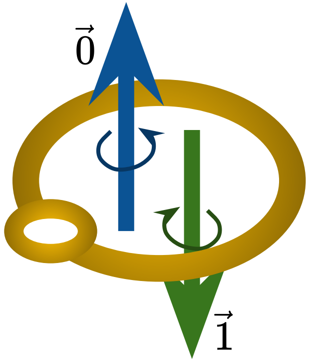


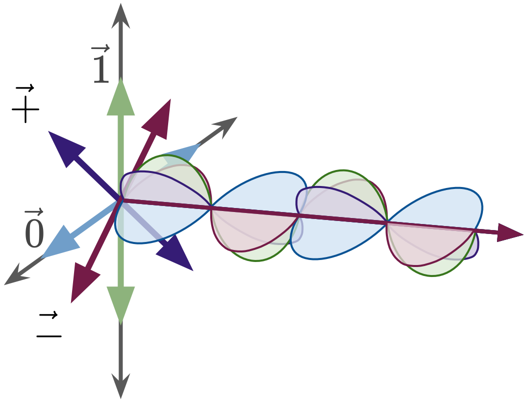

The **qubit** is the basic unit of **quantum information**.

It is a quantum state on **any two-level system**.

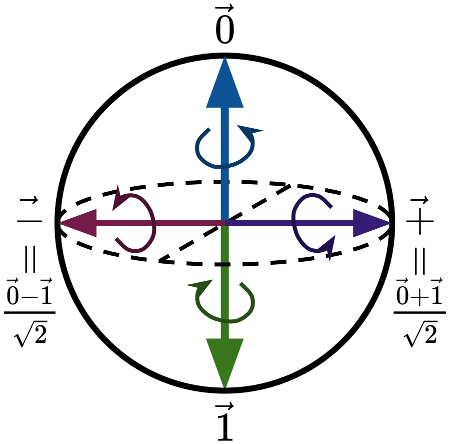
Barak Nehoran (Postdoc, CS Department, Columbia University)

# Examples of Qubits: *Qubits are everywhere!*

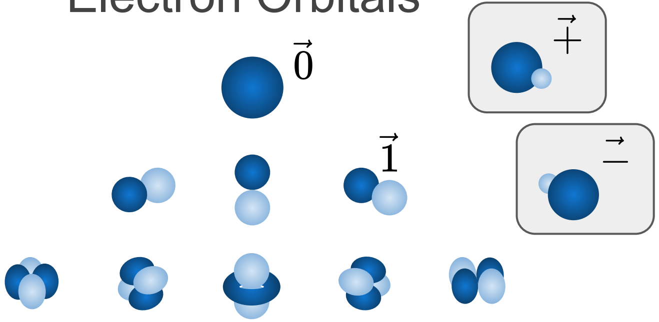
### Light Polarization




### Electron Spin




### Electron Orbitals

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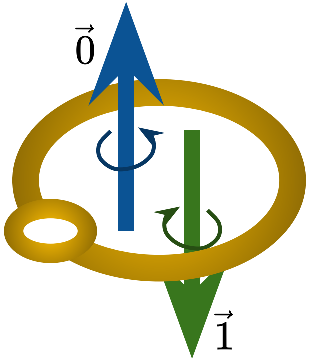


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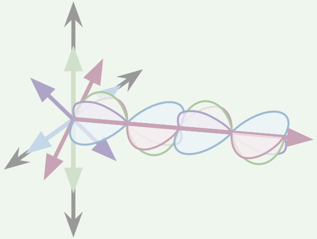
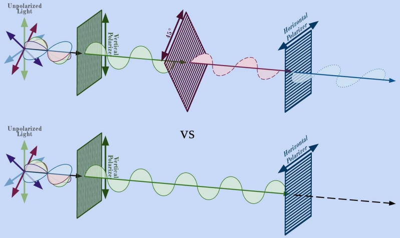
**We abstract a qubit as a unit vector on a 2D space.**

# Outline

1 The Qubit



2 Quantum Measurement

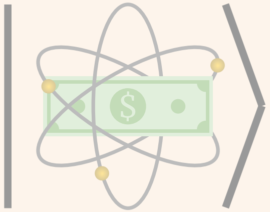



3 Quantum Entanglement & Many Qubits

First qubit Second qubit

$$\frac{1}{\sqrt{2}}|0\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|0\rangle$$

4 No Cloning and Quantum Money



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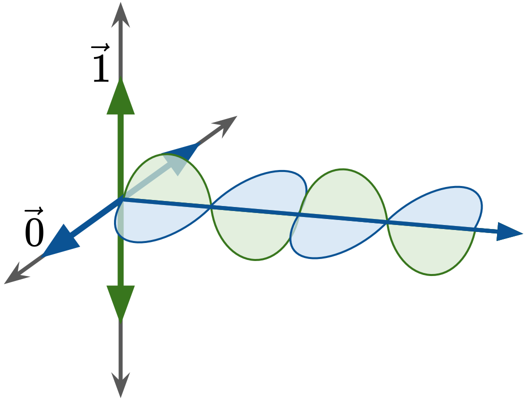
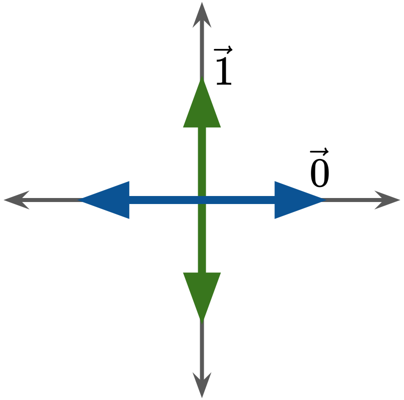
COLUMBIA UNIVERSITY IN THE CITY OF NEW YORK

# Quantum Measurement

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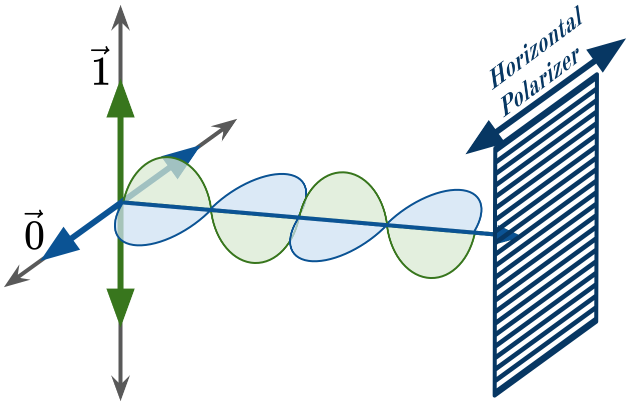


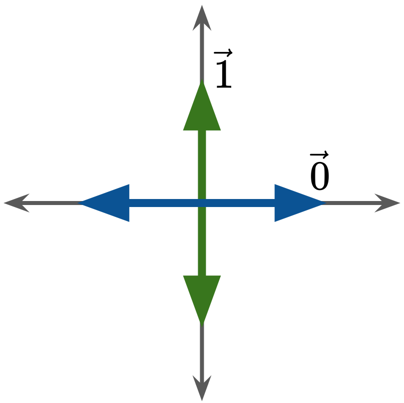


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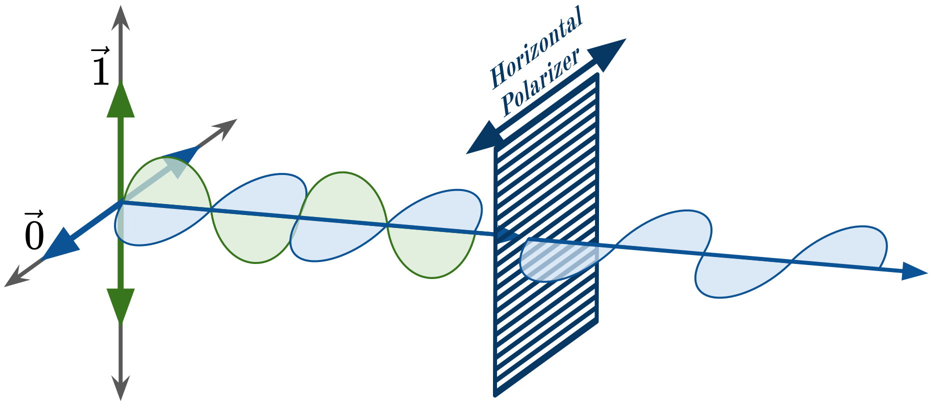


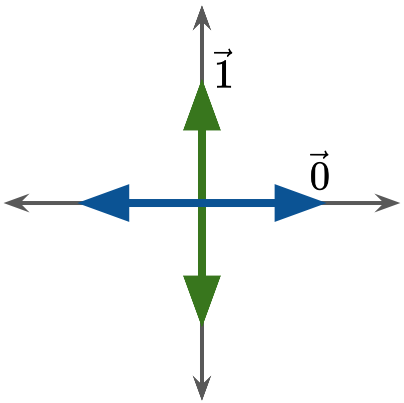


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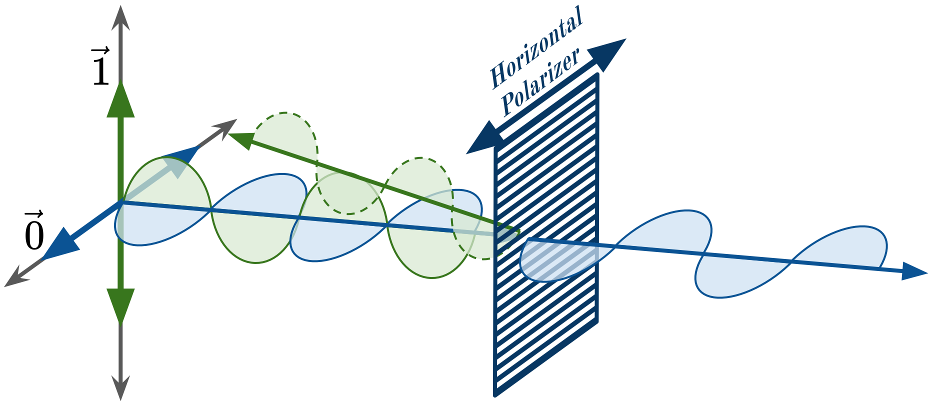


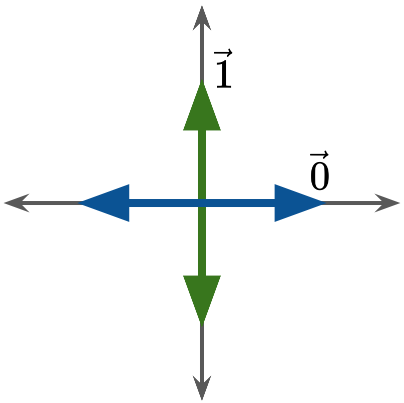


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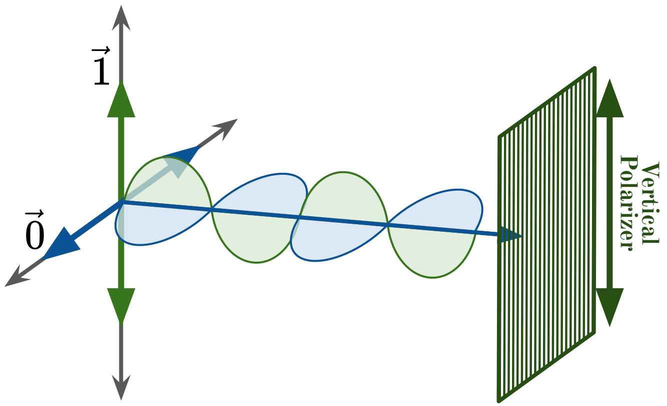


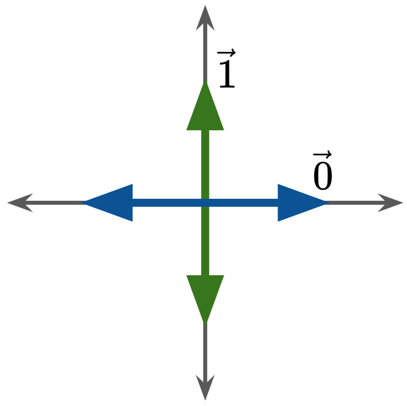


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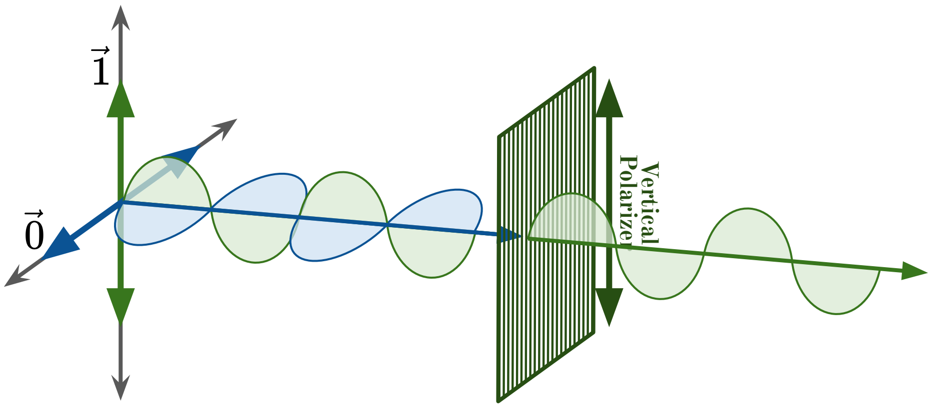


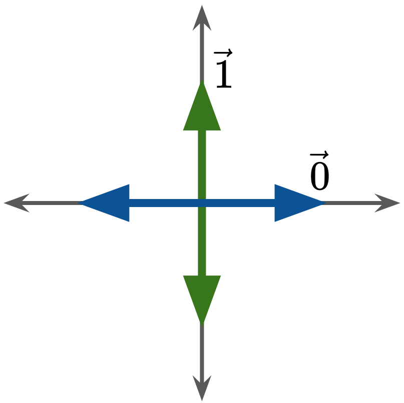


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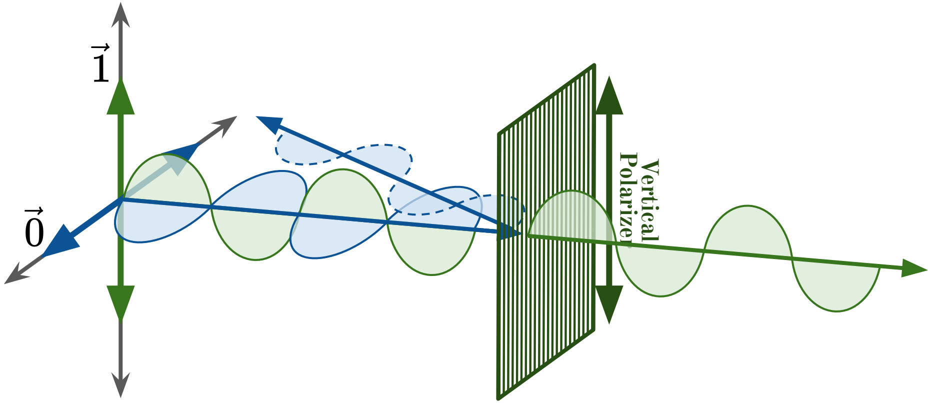


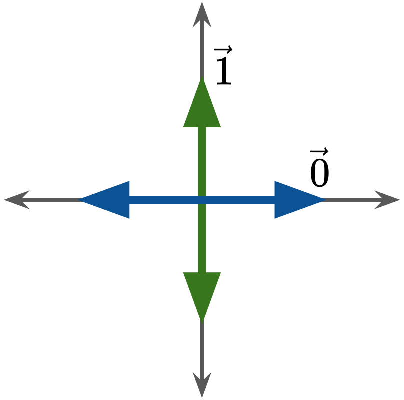


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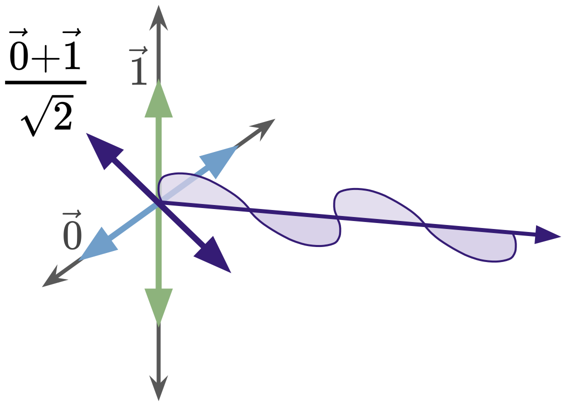
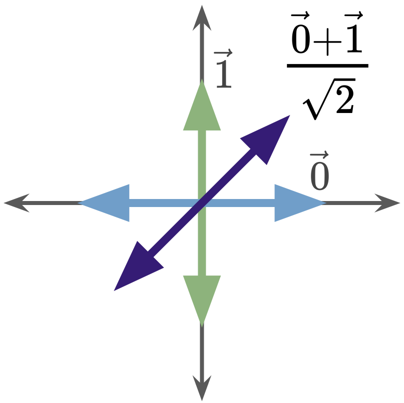
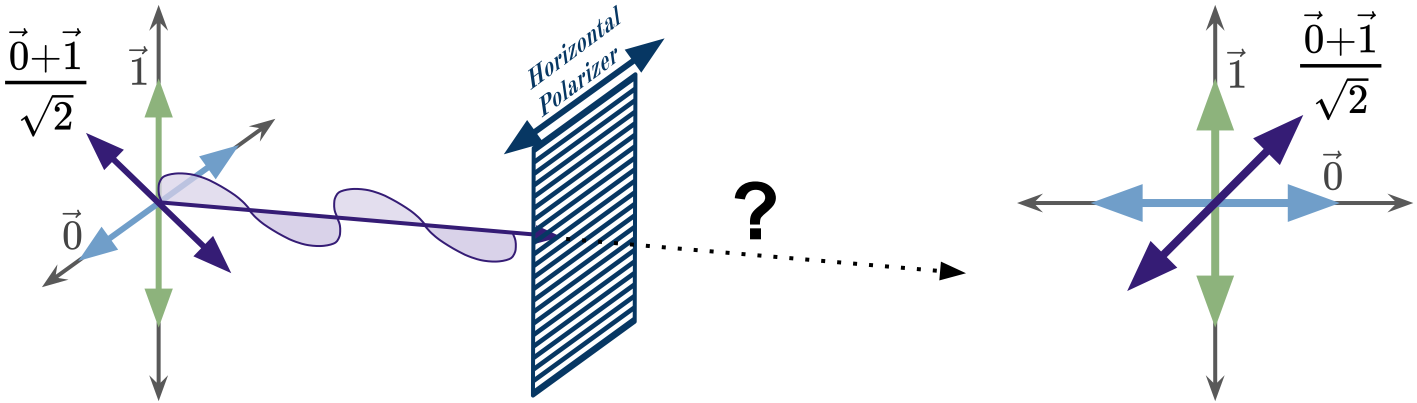



What happens if we send a **single photon** in the **+ state**...

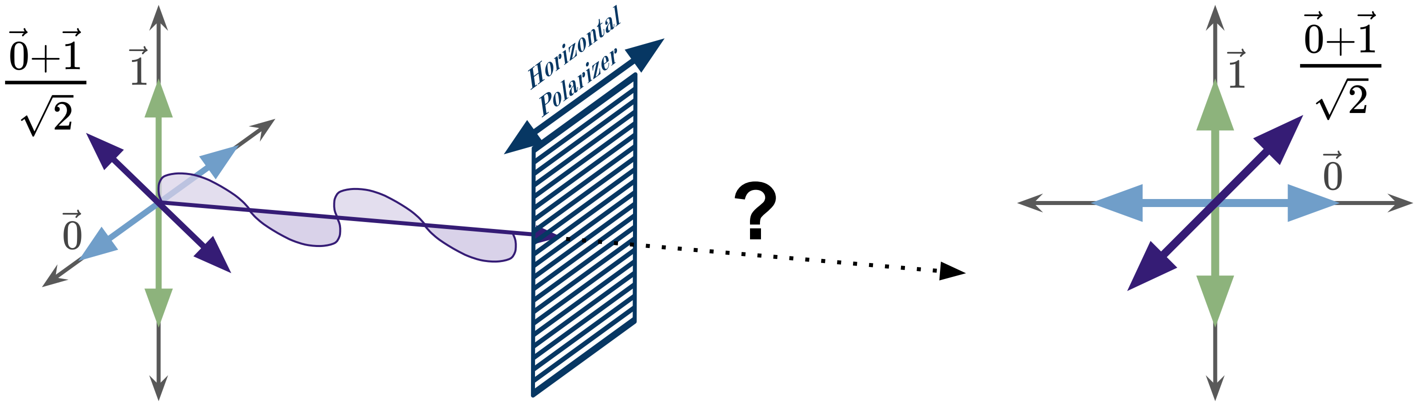
# Quantum Measurement




What happens if we send a **single photon** in the **+ state** through a **horizontal polarizer**?

(a) It **reflects** off the polarizer
(b) It **passes through** the polarizer
(c) It changes its polarization to **vertical**
(d) It changes its polarization to **horizontal**

# Quantum Measurement

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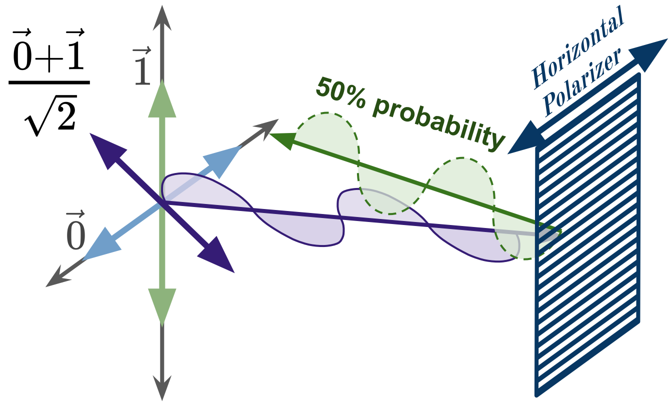


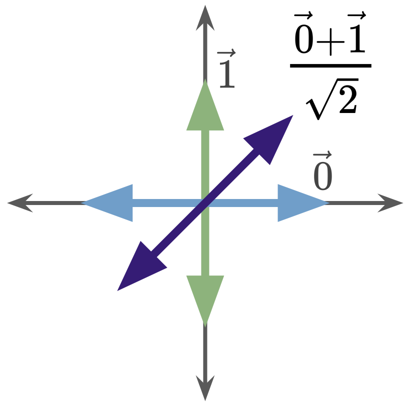


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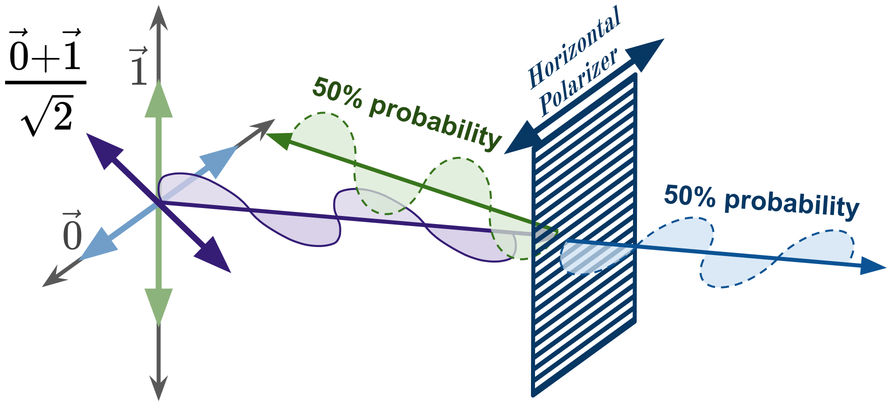
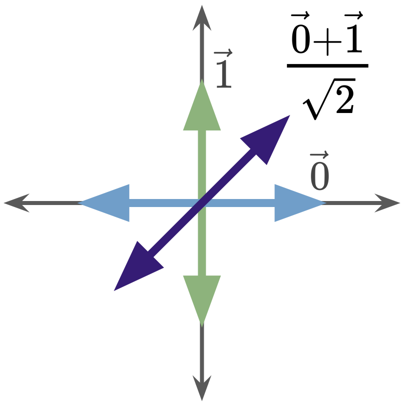
**50% probability**

Barak Nehoran (Postdoc, CS Department, Columbia University)

# Quantum Measurement

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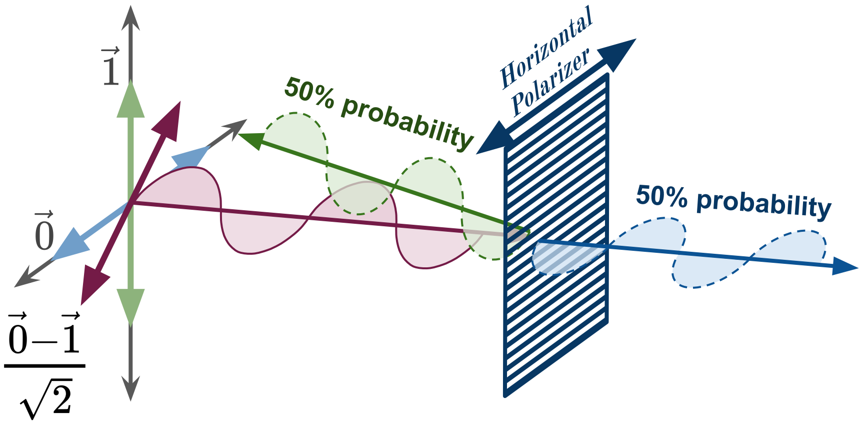
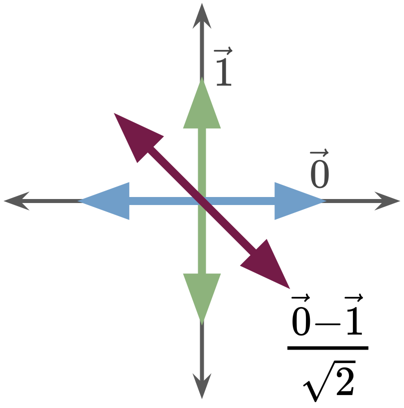



What happens if we send a **single photon** in the **– state** through a **horizontal polarizer**?

(a) It **reflects** off the polarizer

(b) It **passes through** the polarizer

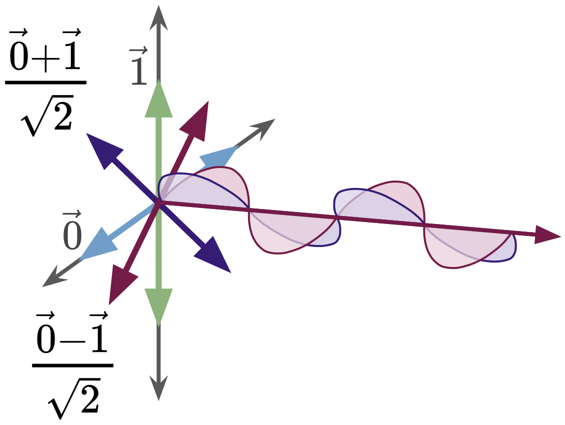
(c) It changes its polarization to **vertical**

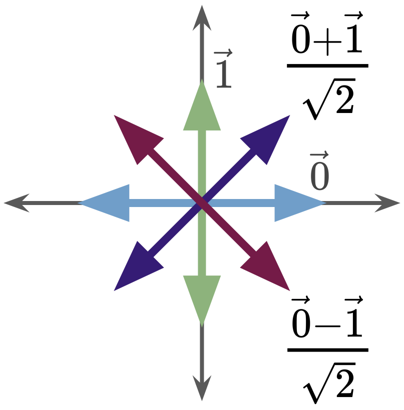
(d) It changes its polarization to **horizontal**

**50% probability**

**50% probability**

# Quantum Measurement



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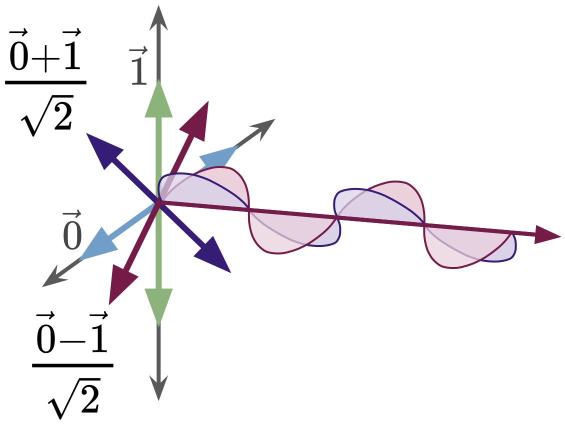
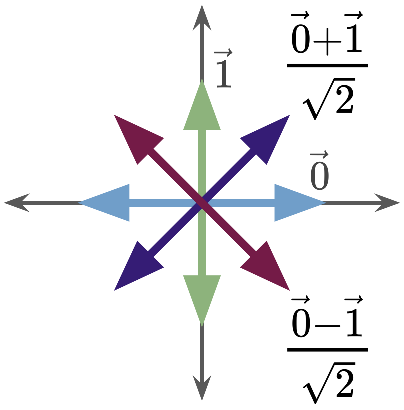
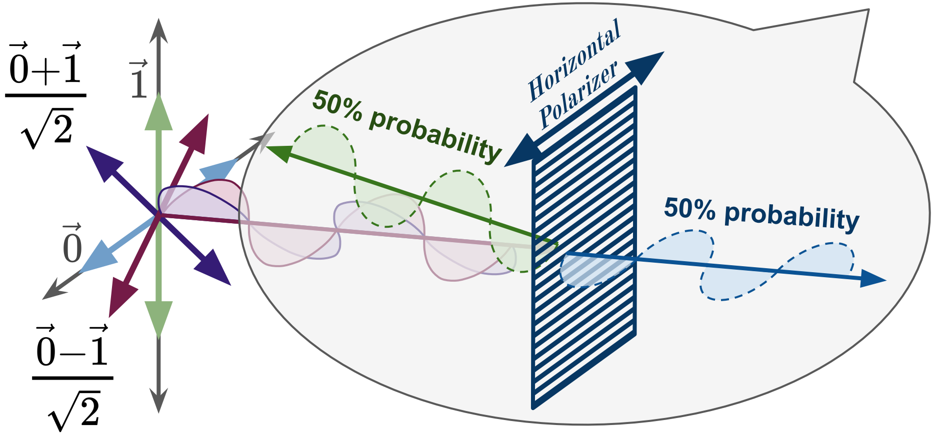


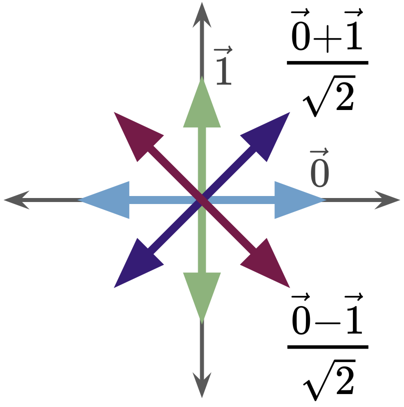




Suppose we have a **single photon** in *either* the **+ state** or the **– state**, and we don't know which.
**Can we distinguish** between the two diagonal polarizations?

# Quantum Measurement

*A horizontal polarizer does not distinguish!*

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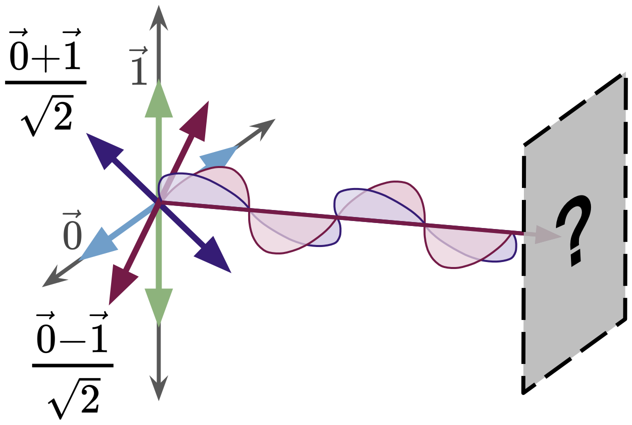


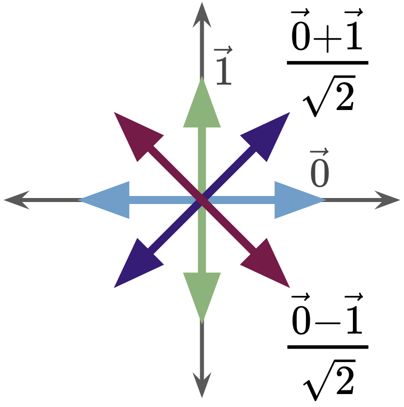


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**Can we distinguish** between the two diagonal polarizations?

(a) **No.** The two diagonals are indistinguishable.

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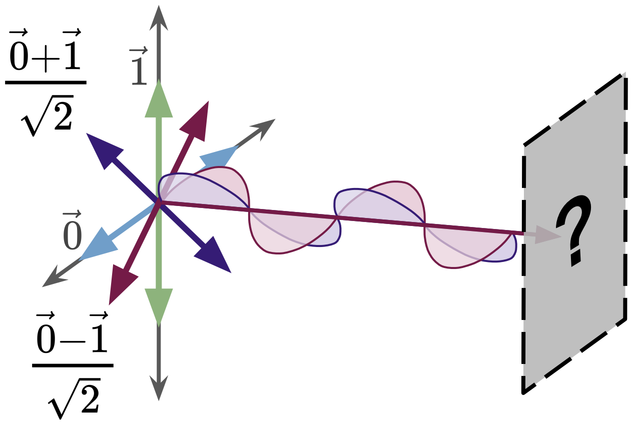


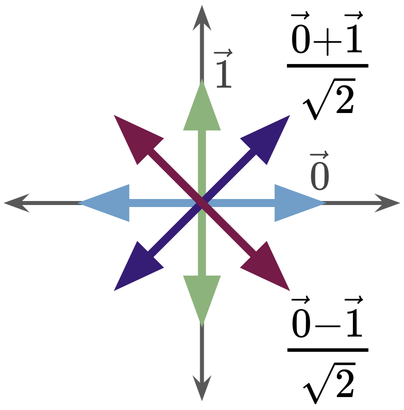


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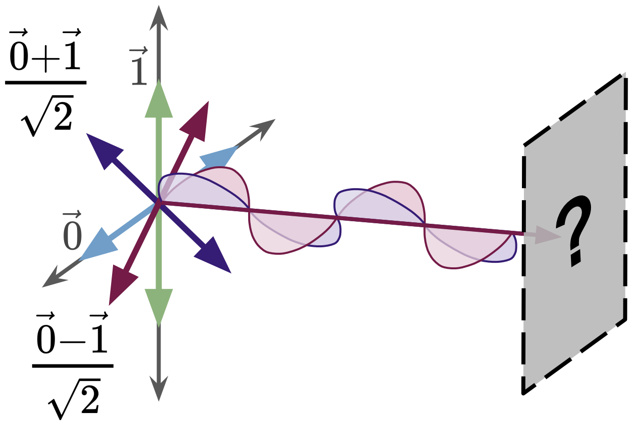


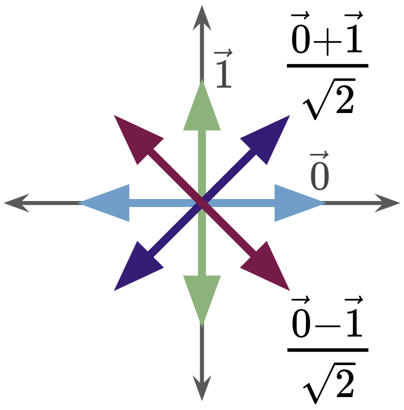


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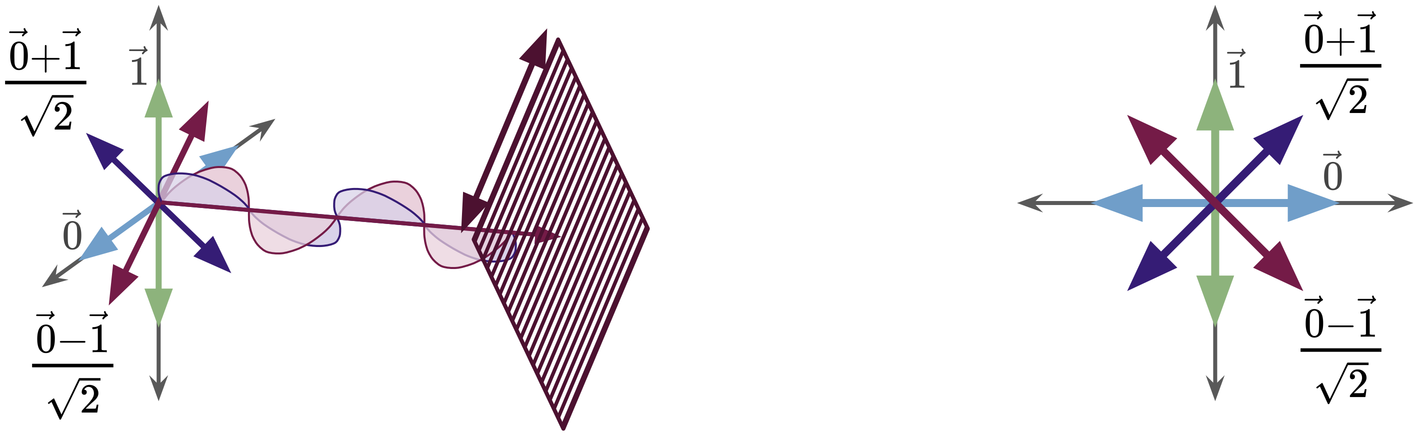


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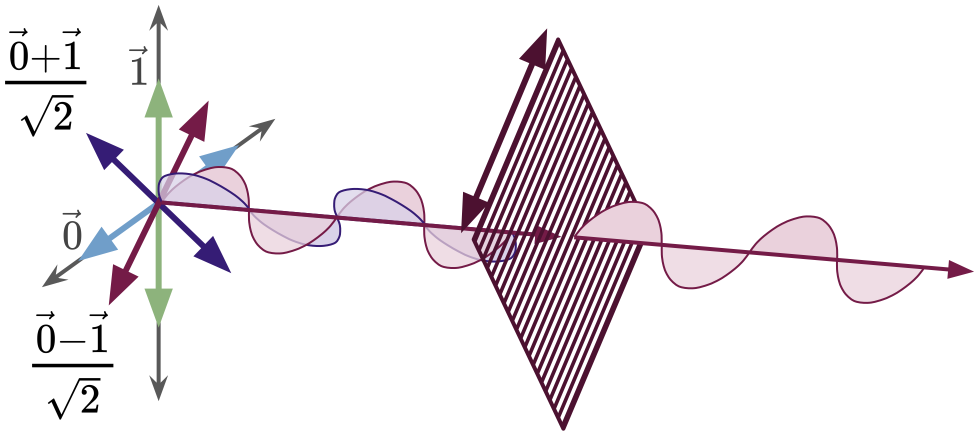


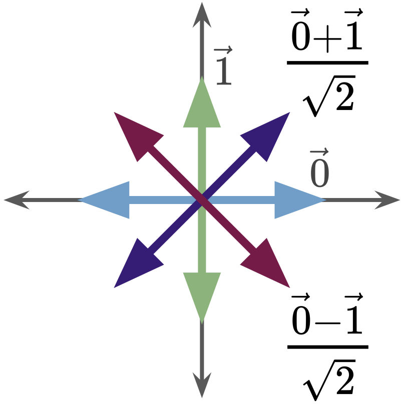


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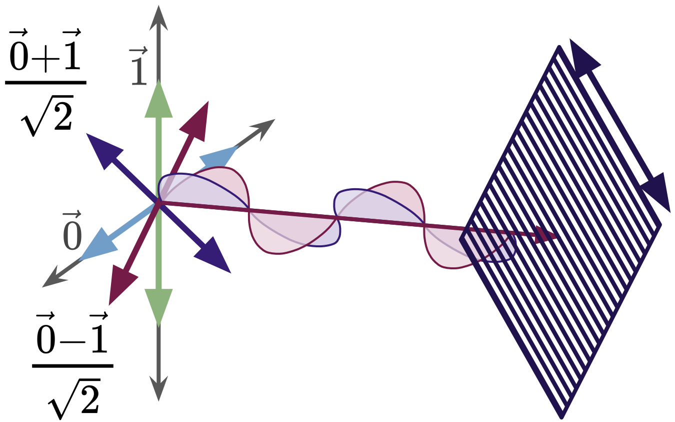


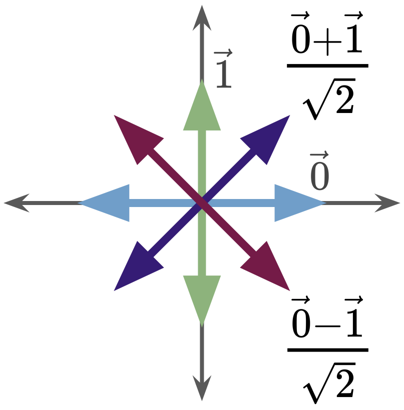


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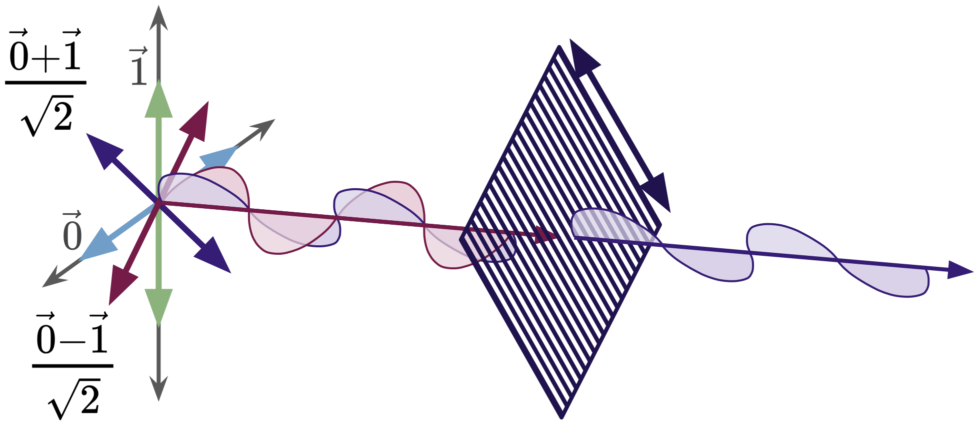


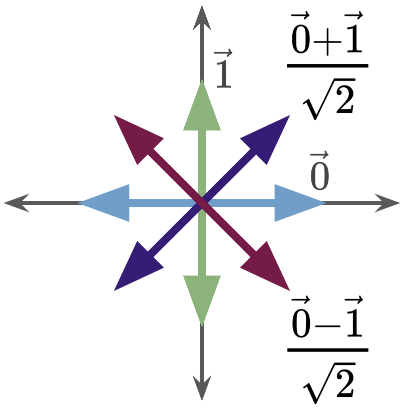


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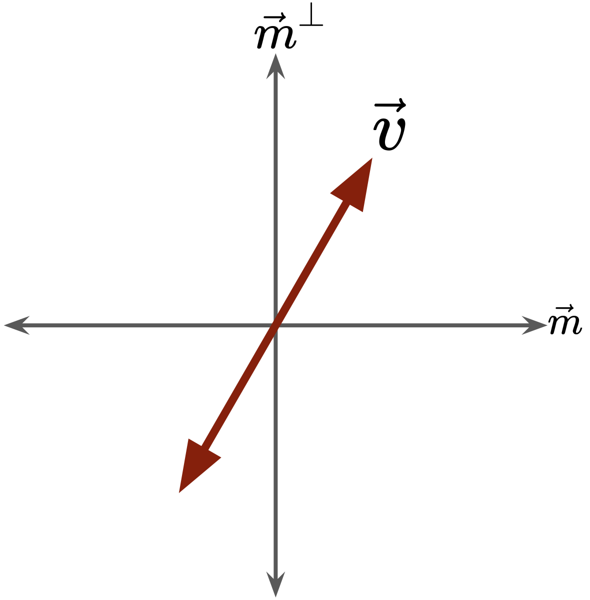
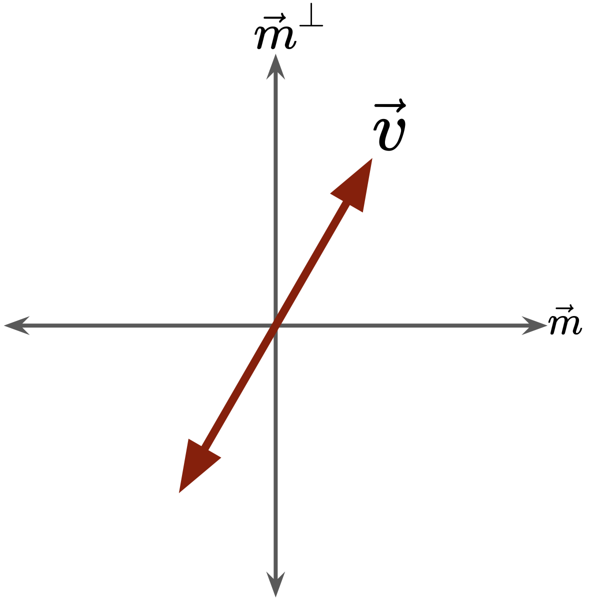



(this view is rotated to depict m along the horizontal)

# Quantum Measurement

**Rule**: Given a state vector $\vec{v}$ , if you measure in direction $\vec{m}$:

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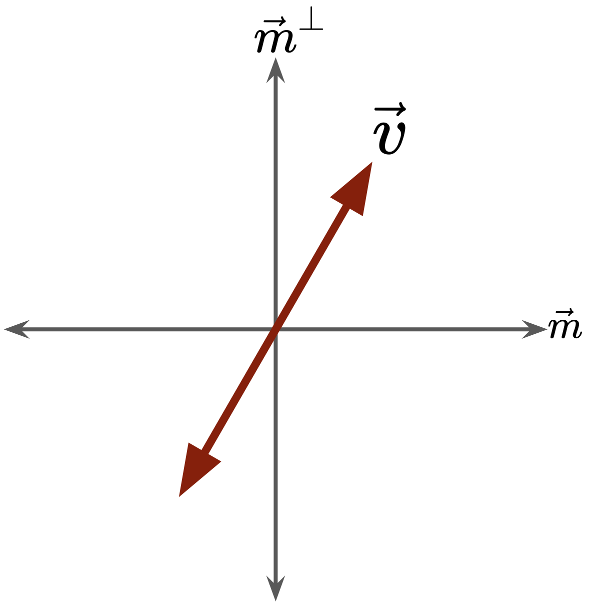
# Quantum Measurement

**Rule**: Given a state vector $\vec{v}$ , if you measure in direction $\vec{m}$:

Project $\vec{v}$ onto $\vec{m}$ and its perpendicular subspace $\vec{m}^{\perp}$

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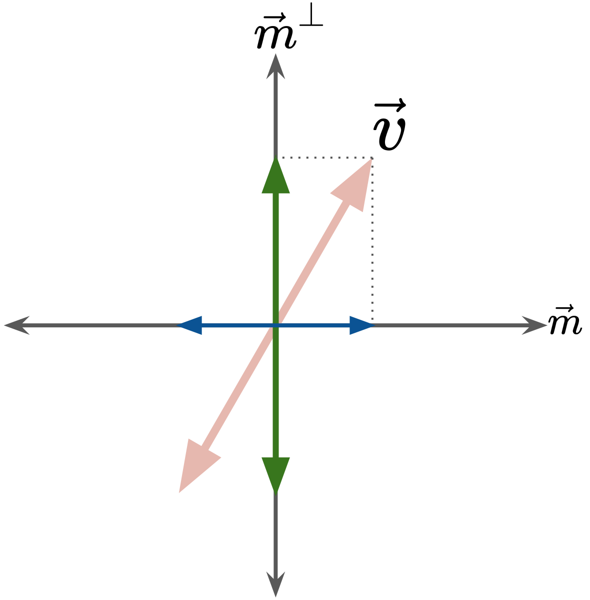


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Barak Nehoran (Postdoc, CS Department, Columbia University)

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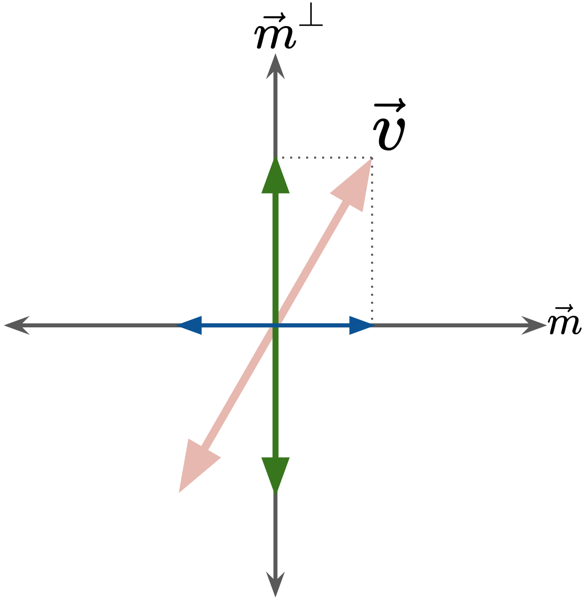

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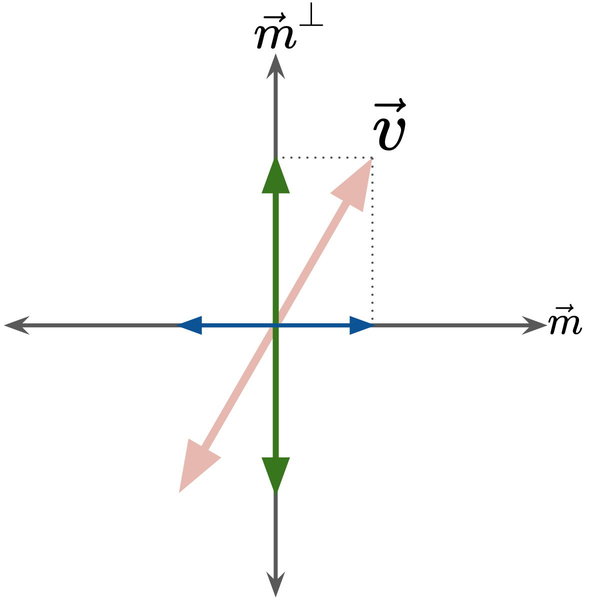


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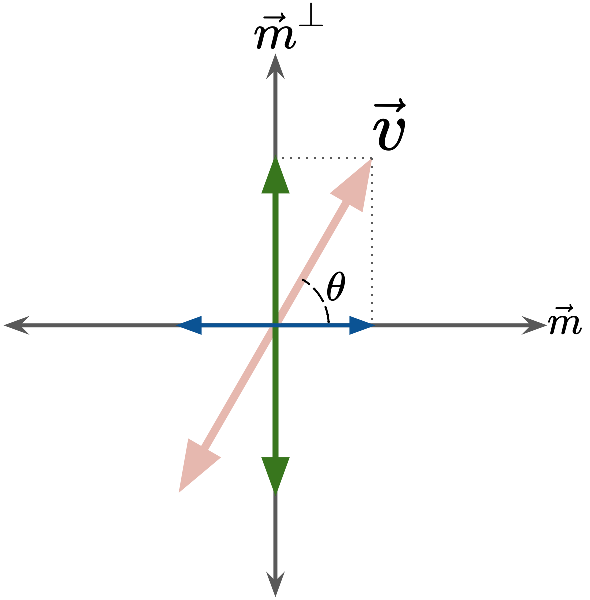


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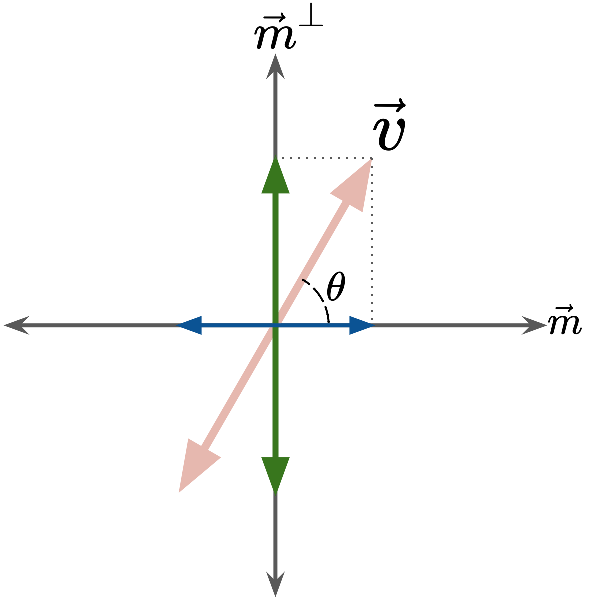

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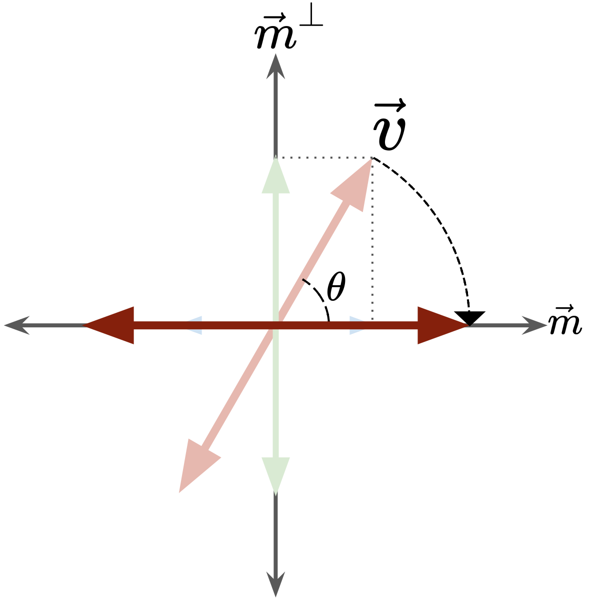

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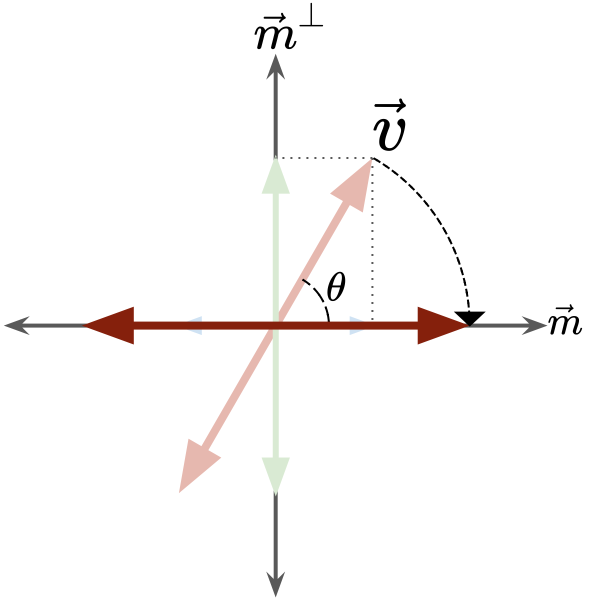

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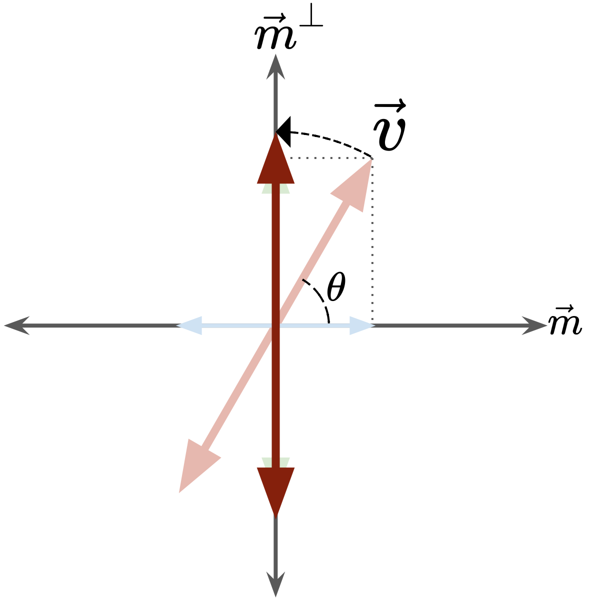


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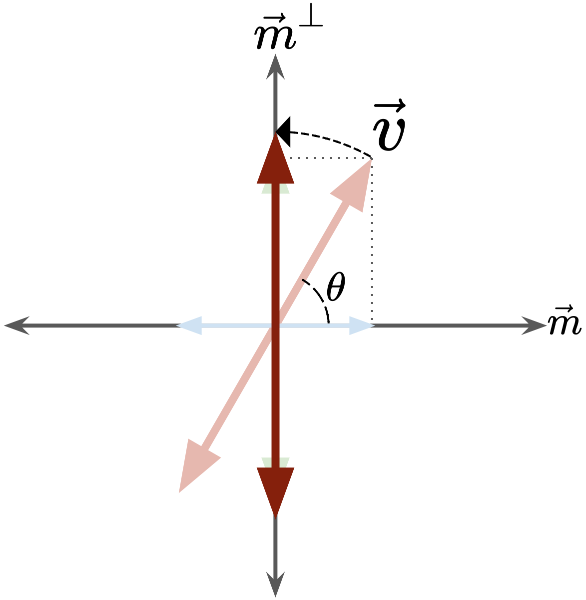

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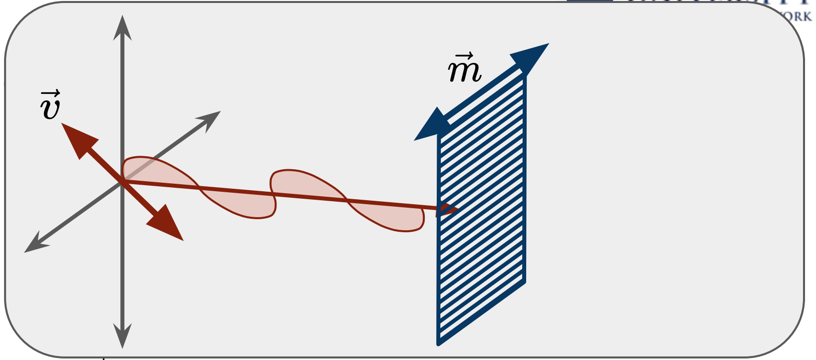


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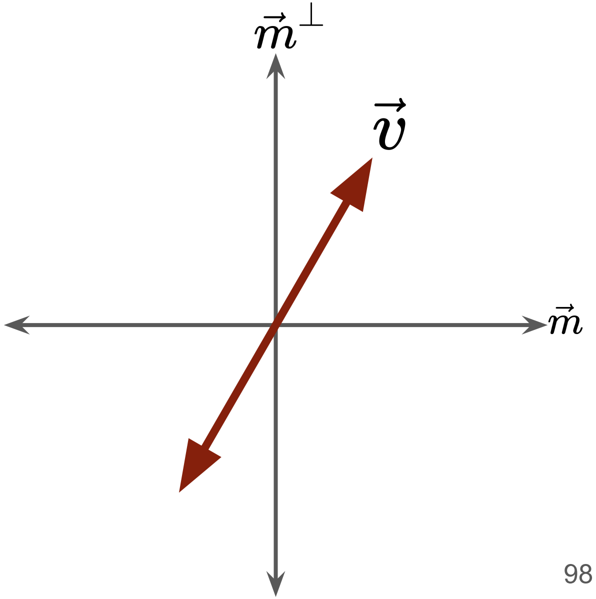

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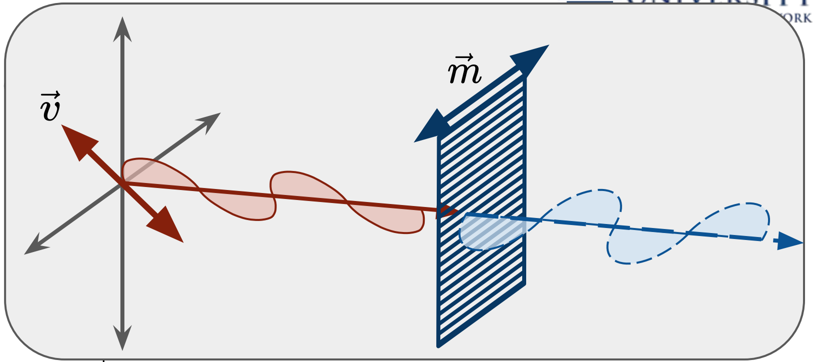


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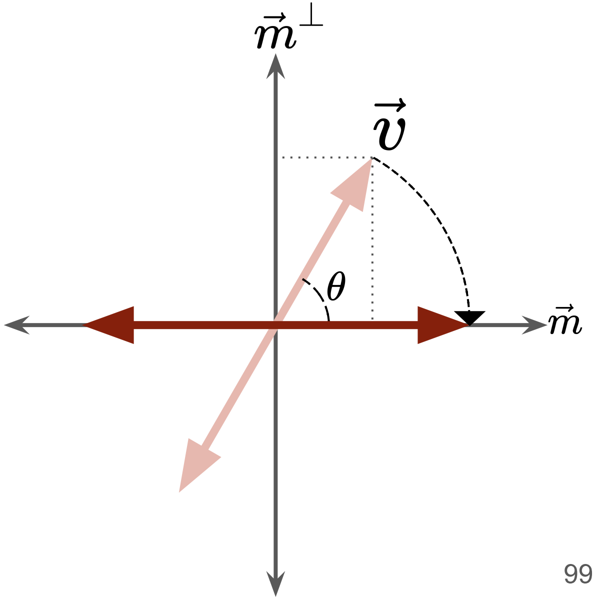

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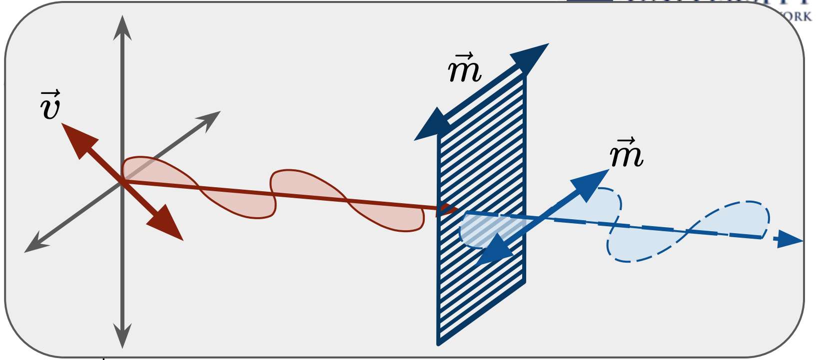


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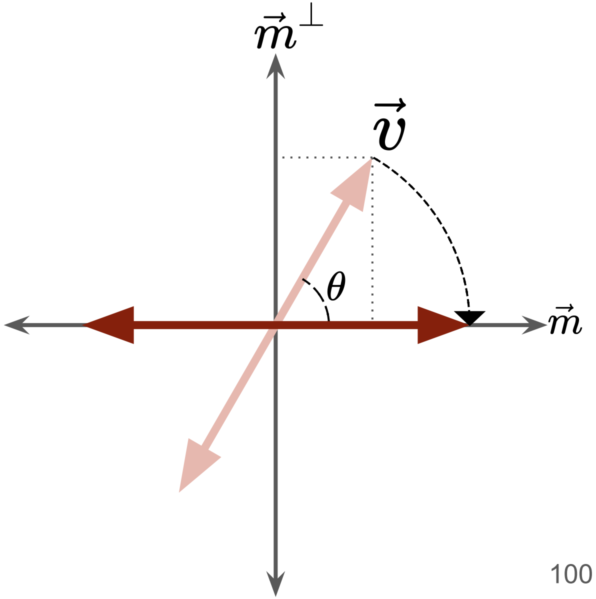

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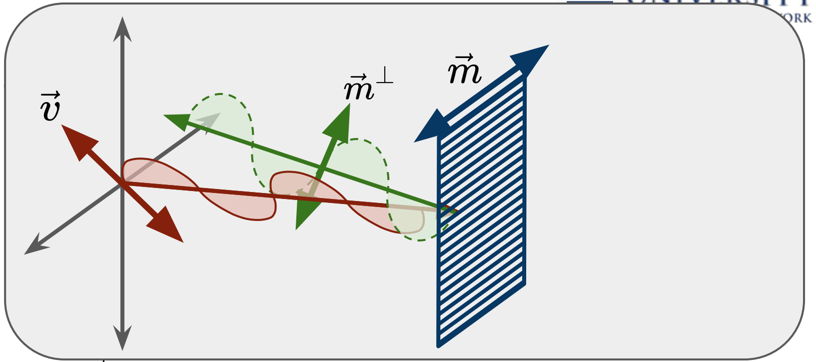


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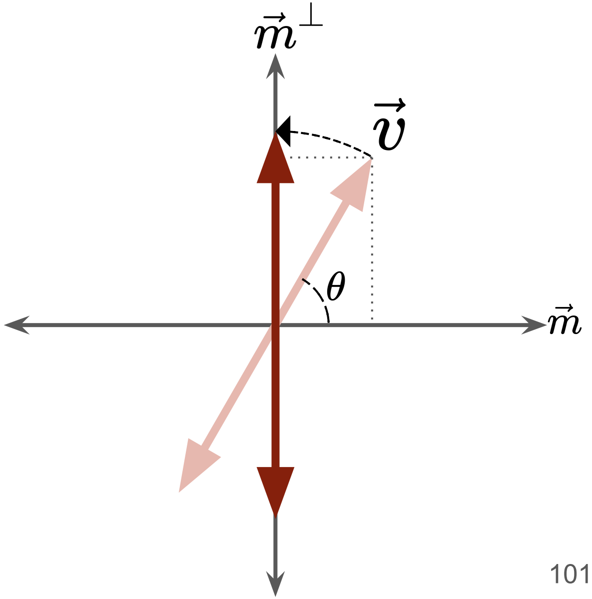

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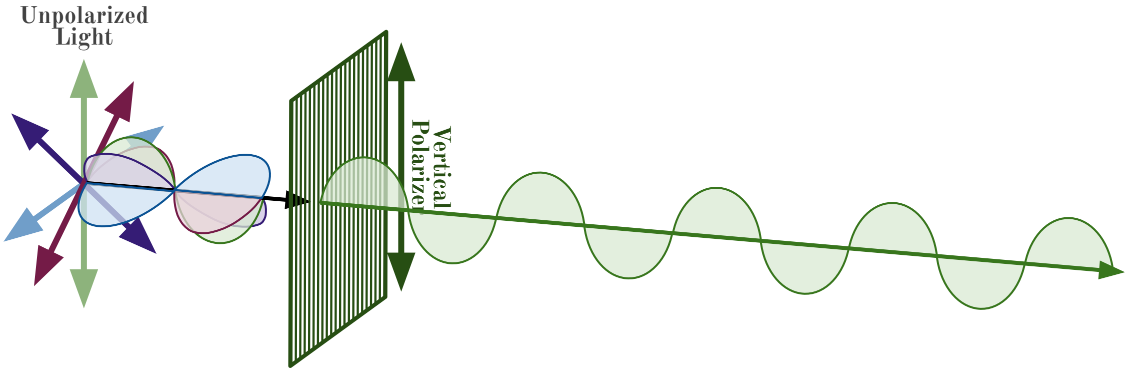

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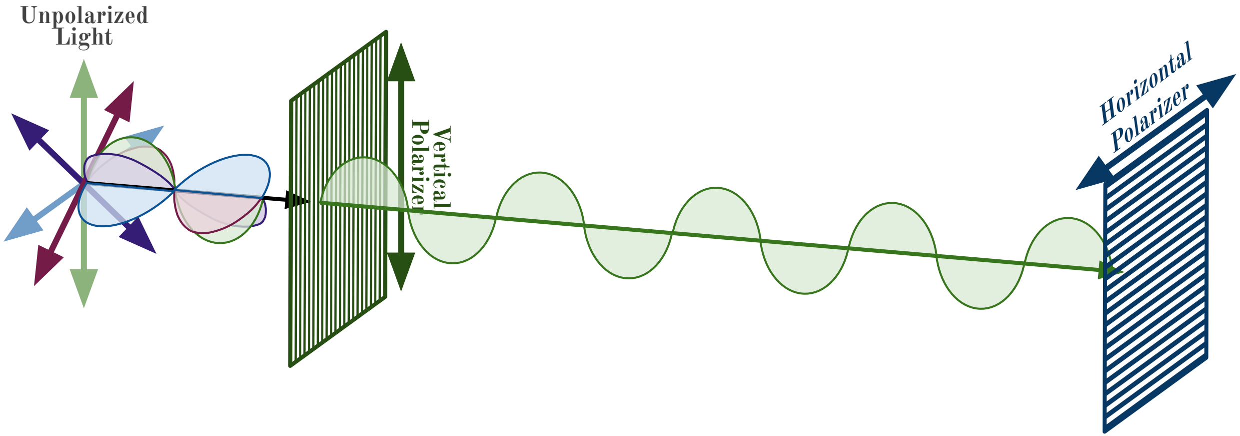

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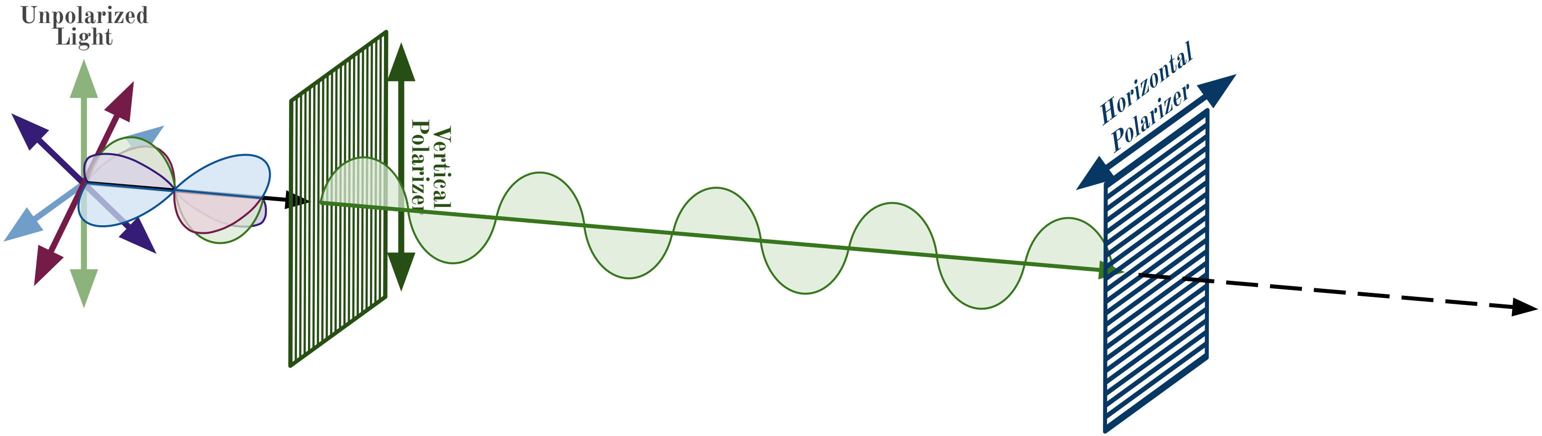

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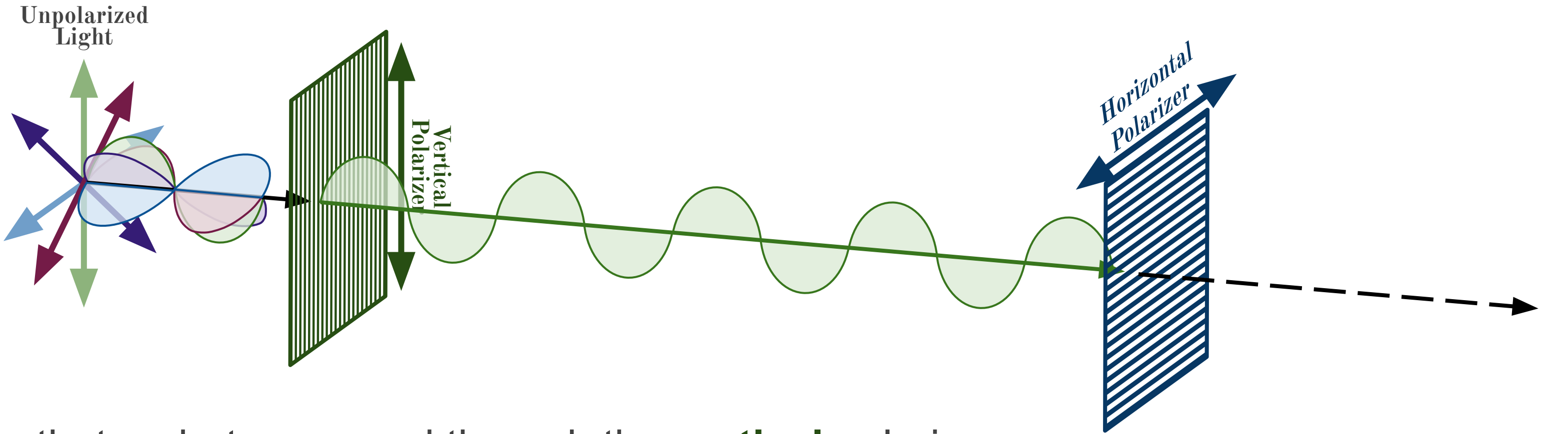


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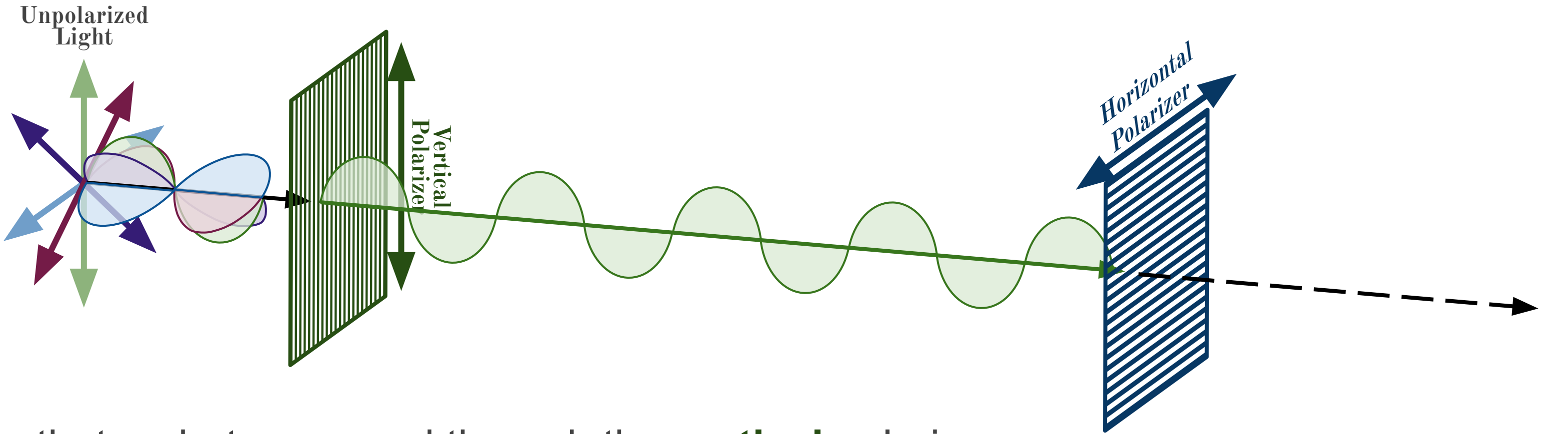


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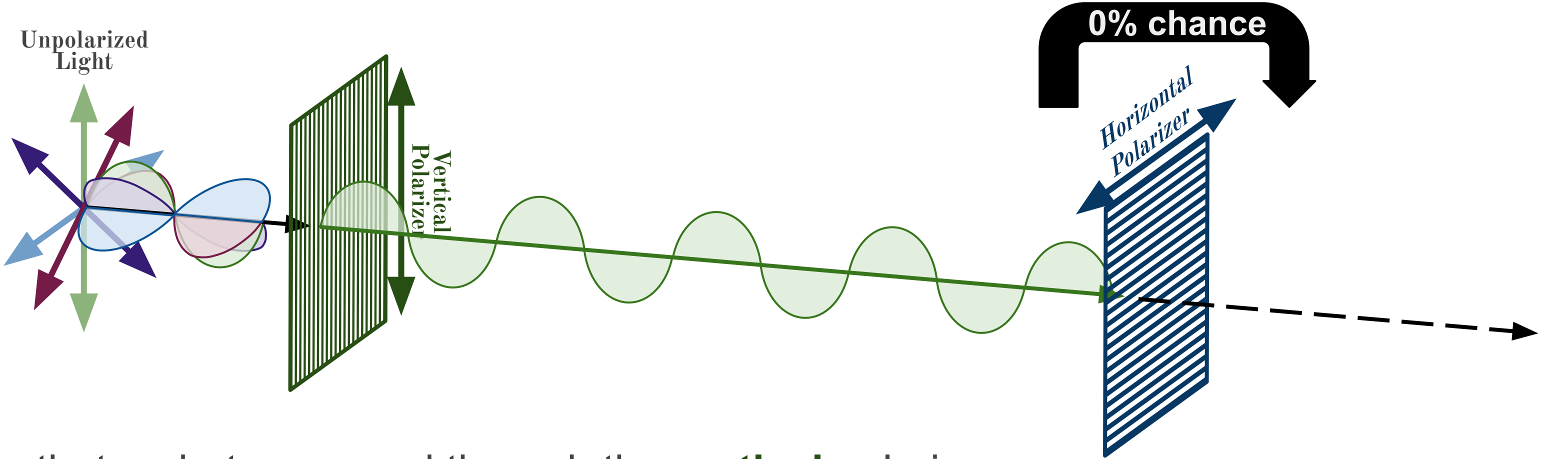


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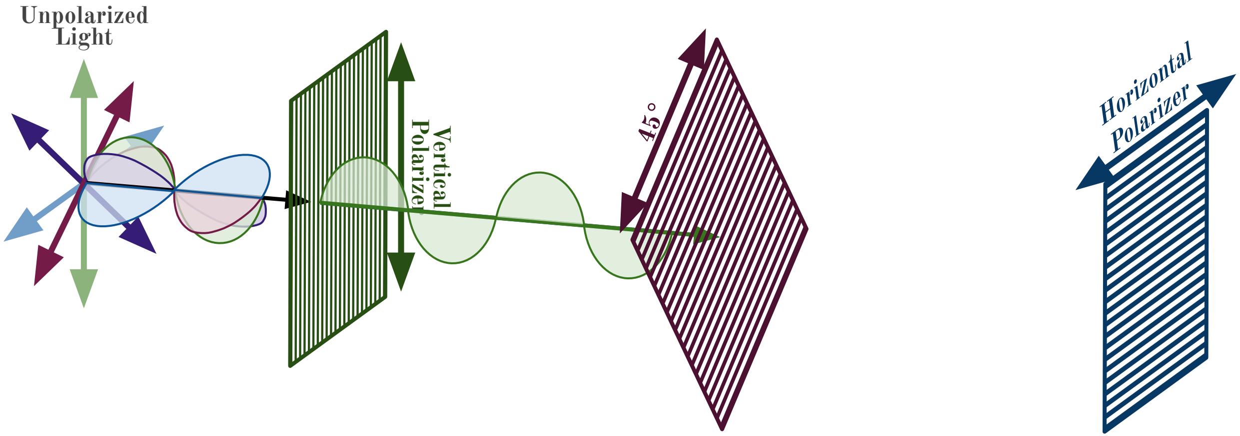

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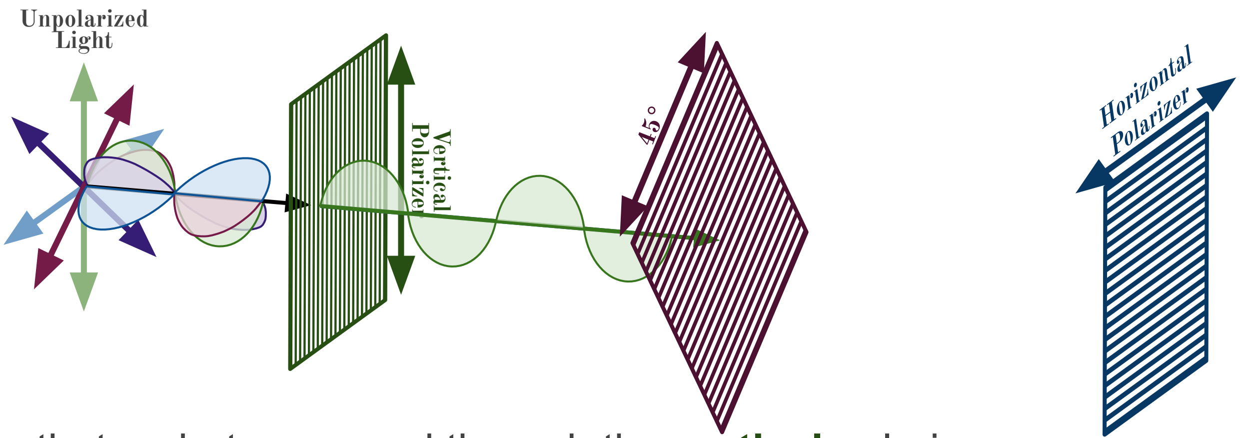


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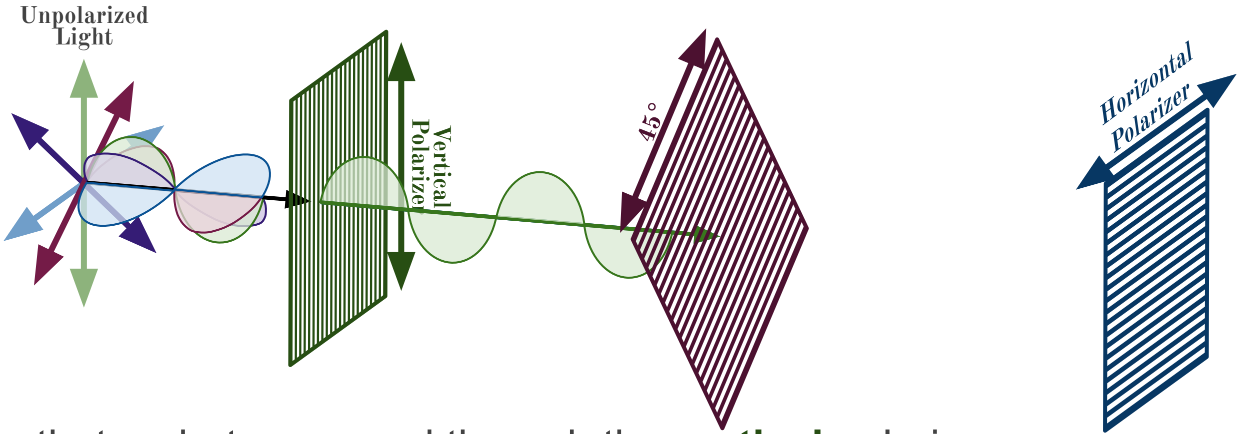


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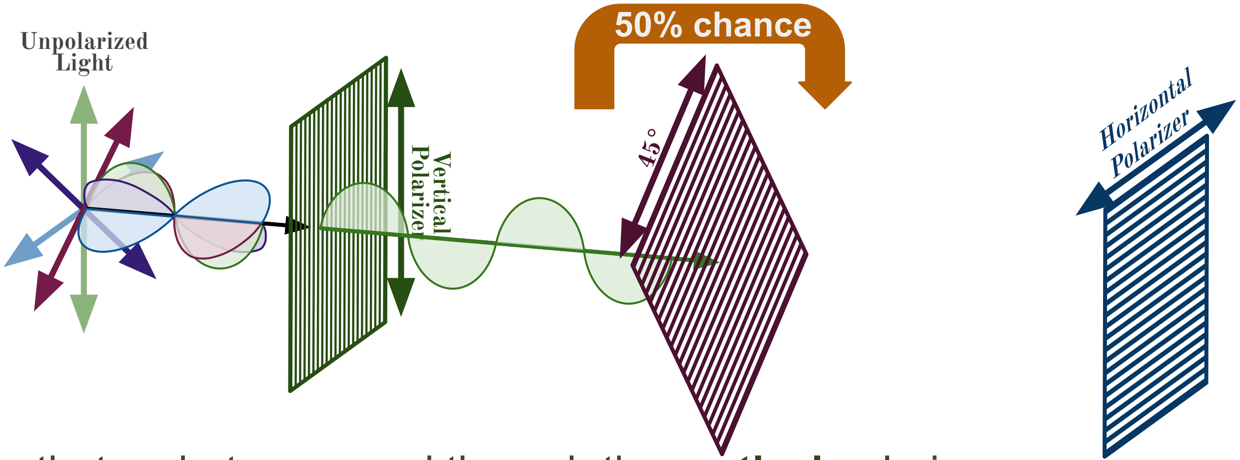


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Columbia University in the City of New York

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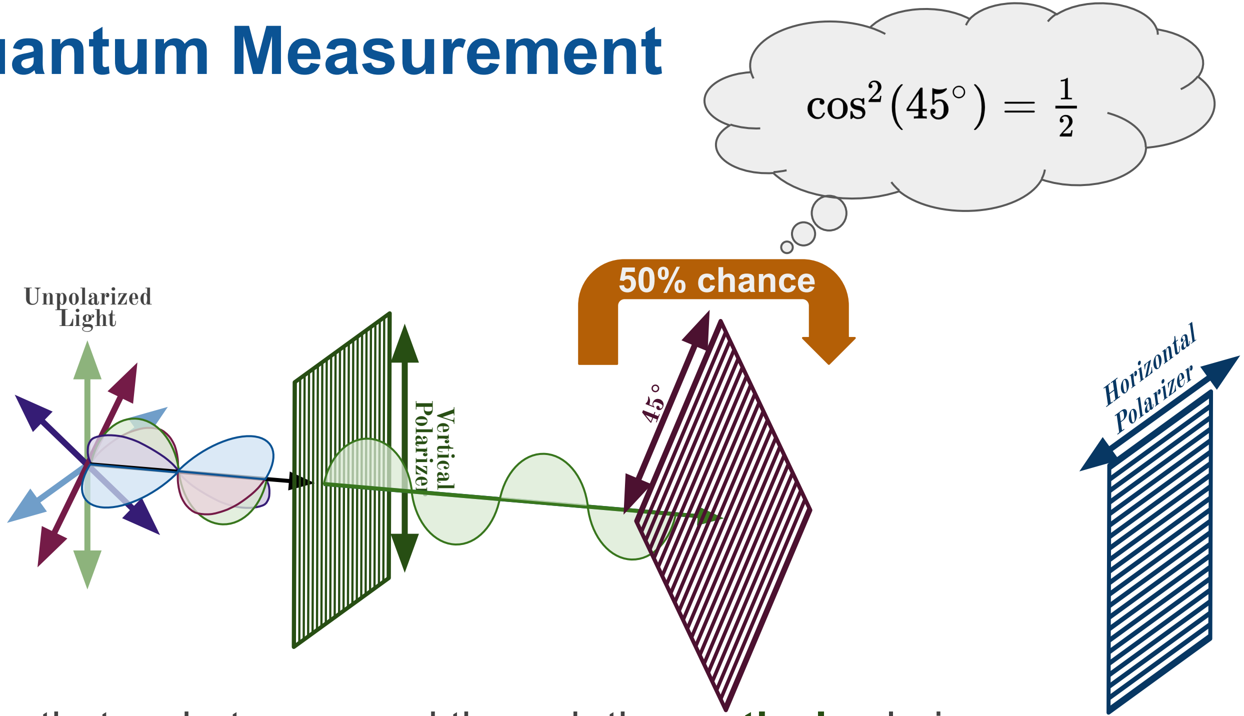


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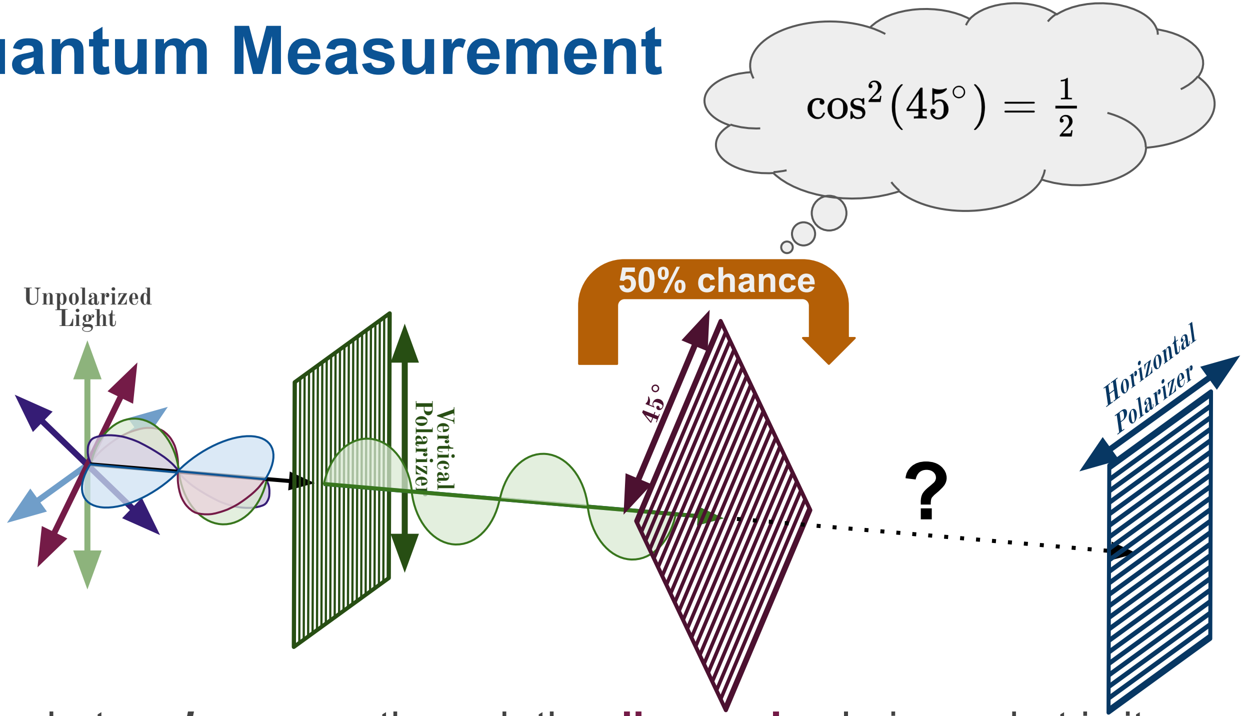


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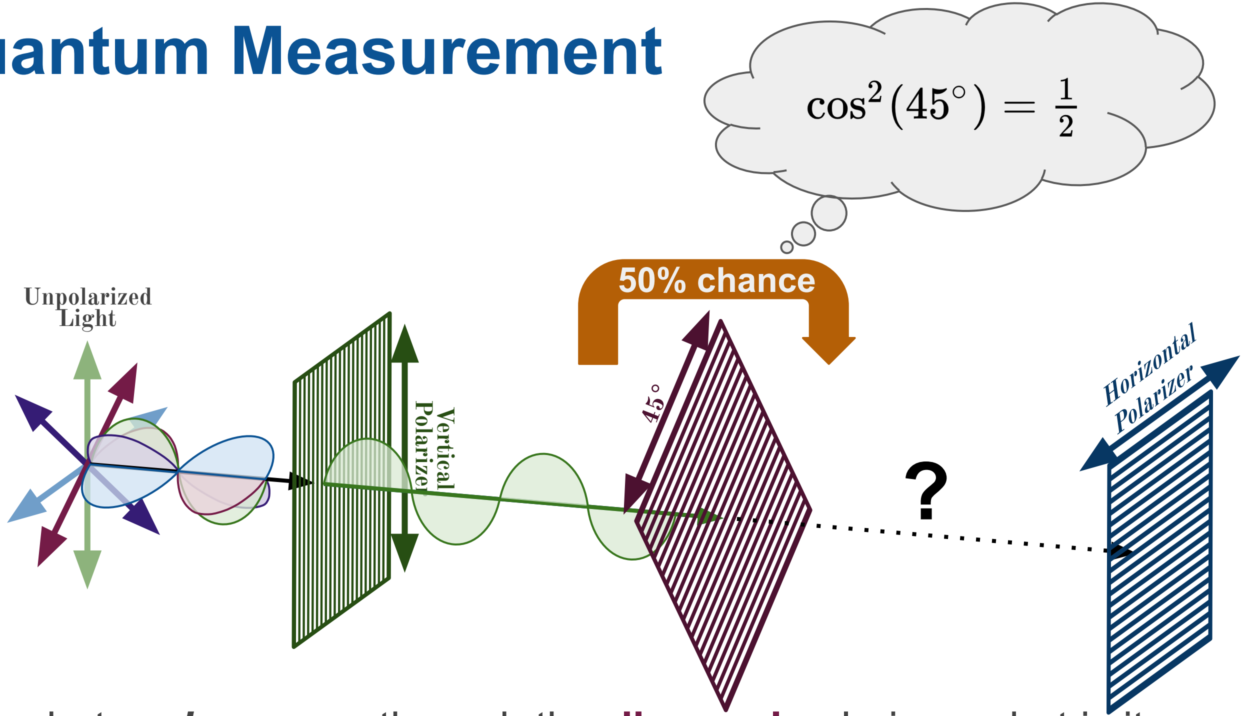


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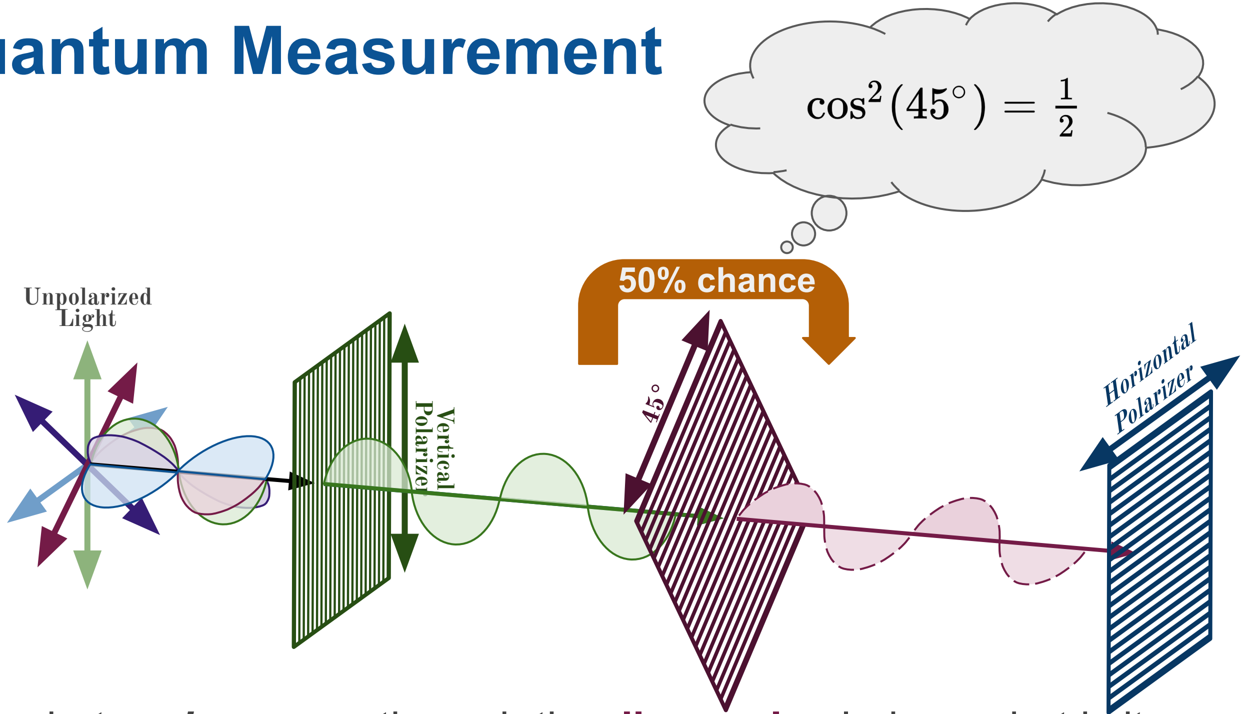


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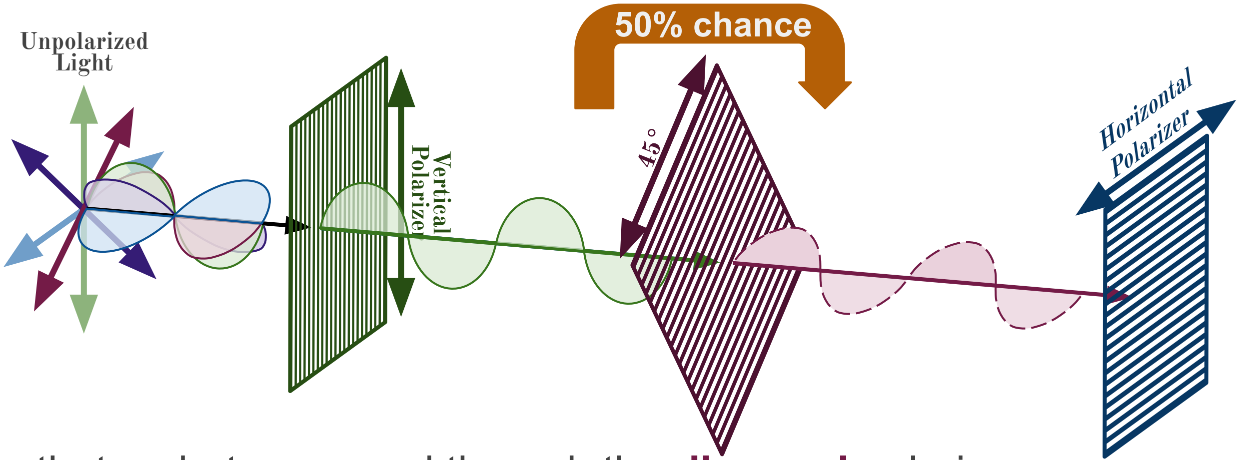


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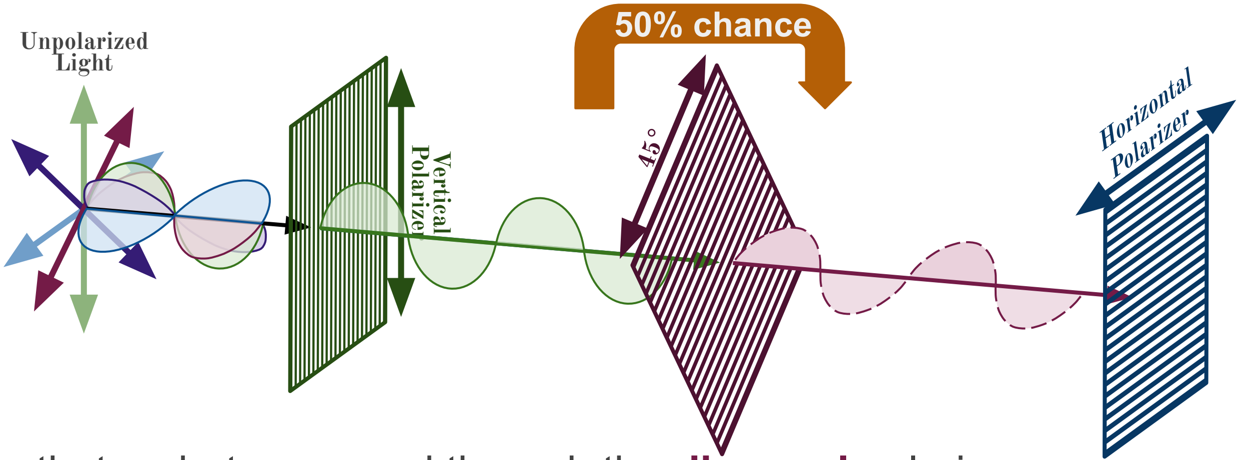


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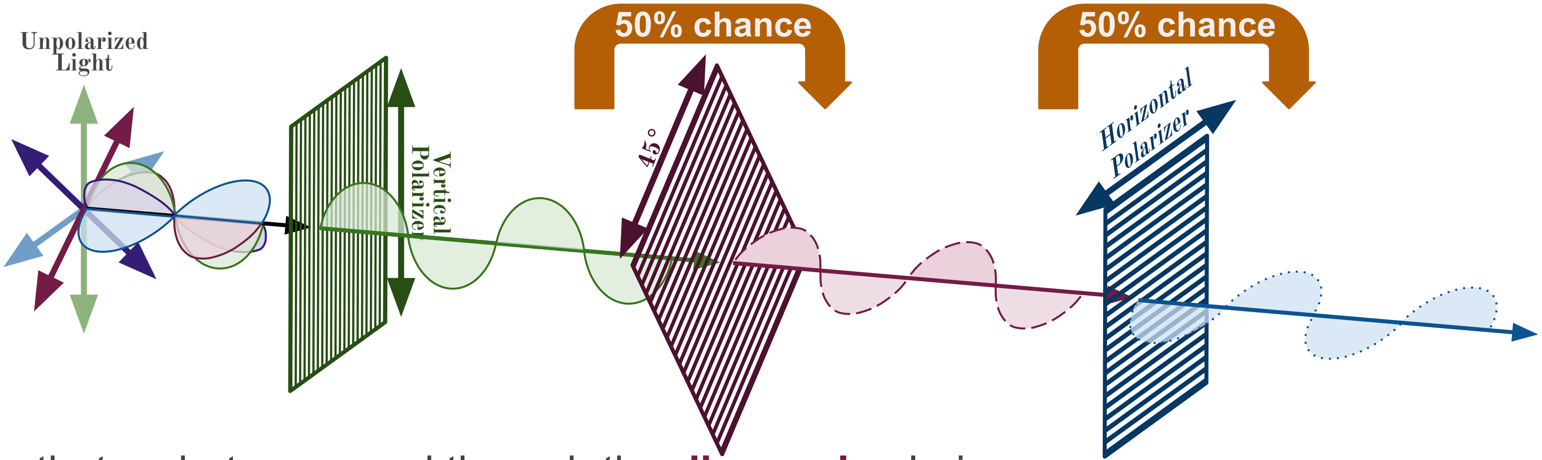


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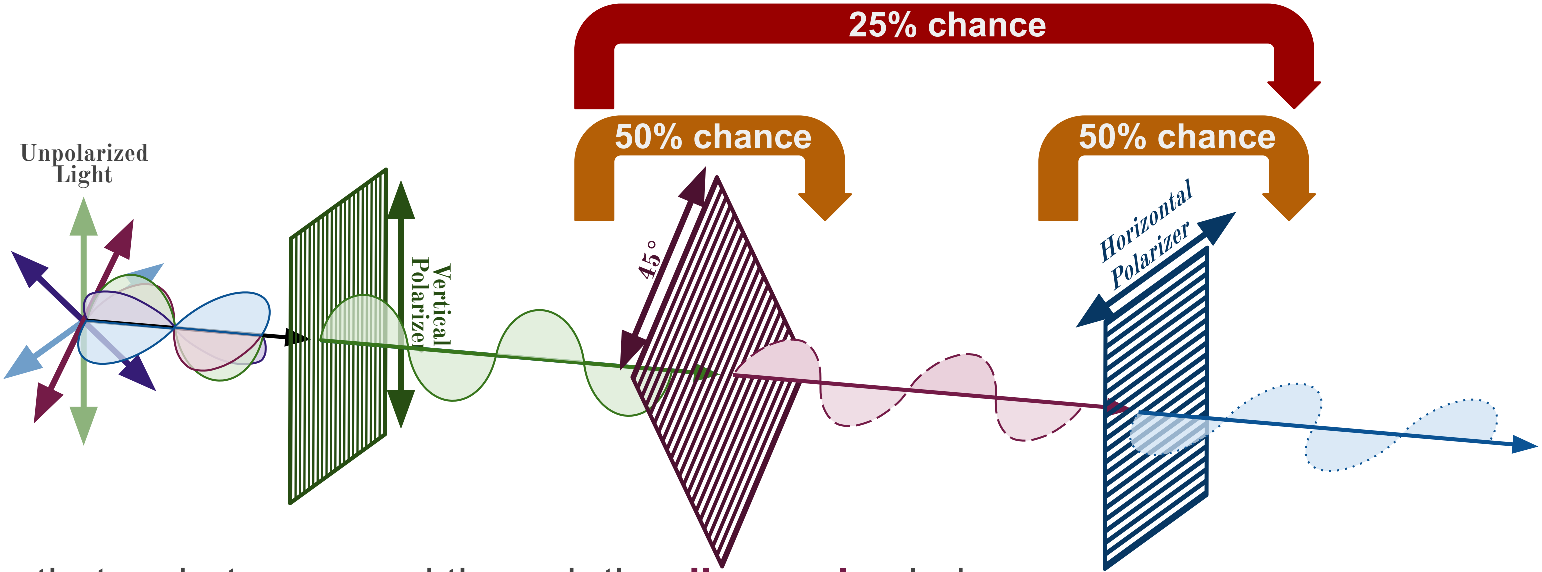


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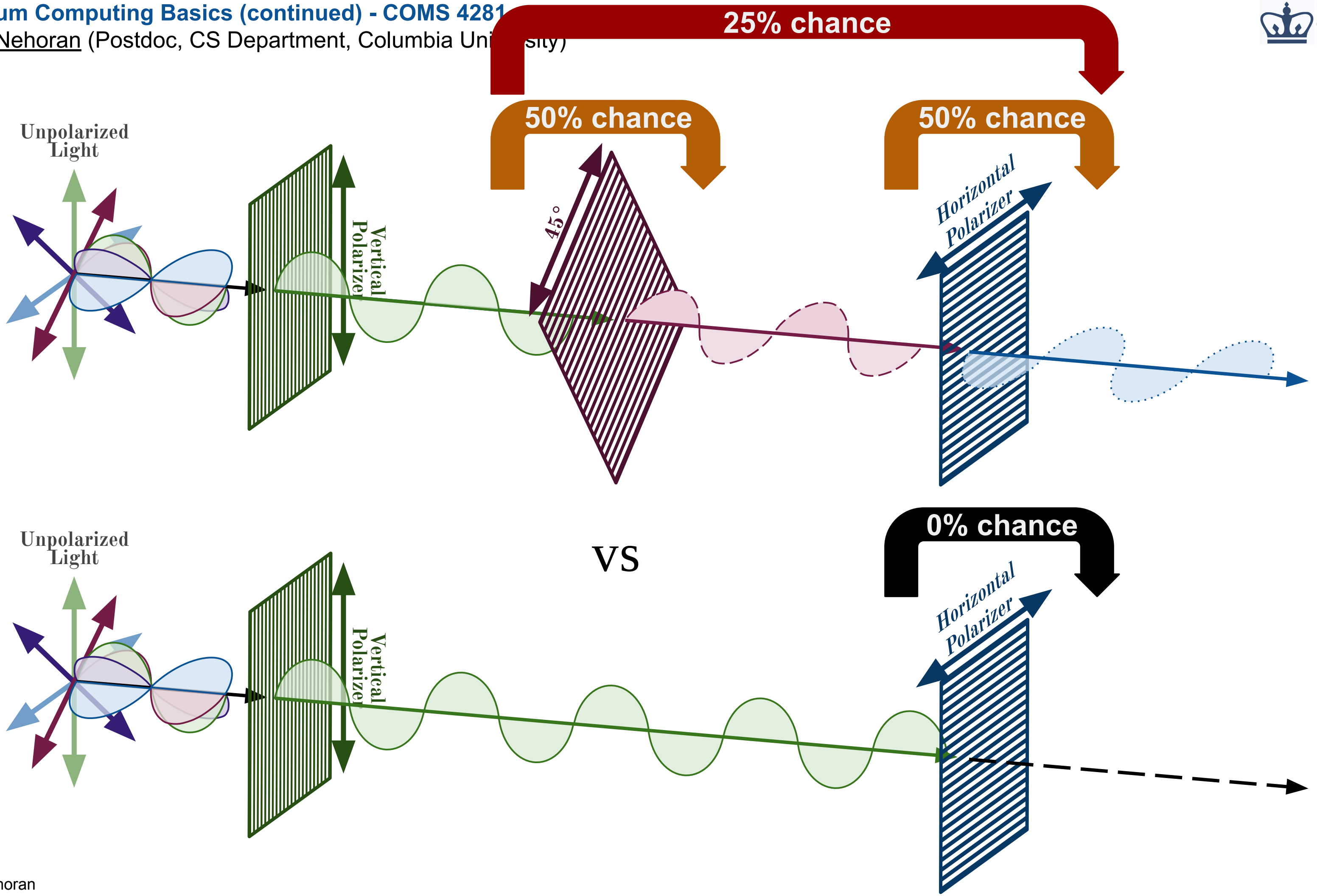
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50% chance
50% chance
Unpolarized Light
Vertical Polarizer
45°
Horizontal Polarizer
VS
0% chance
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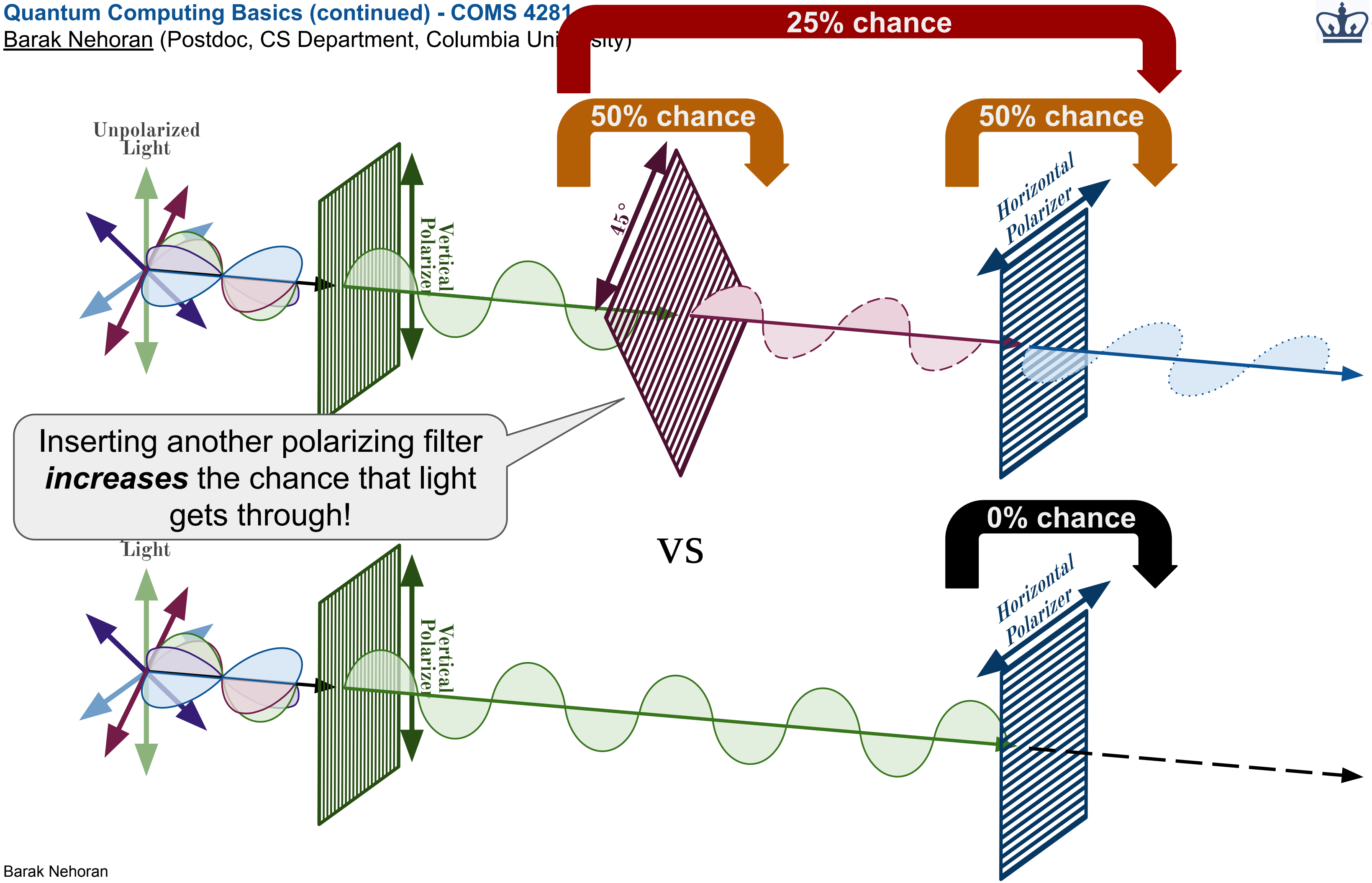

# Unitaries: Operations that don't measure

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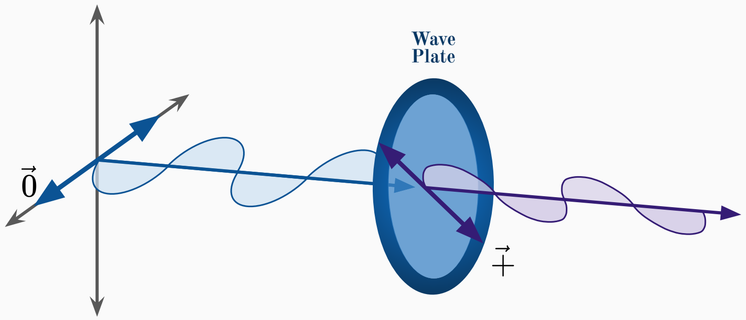

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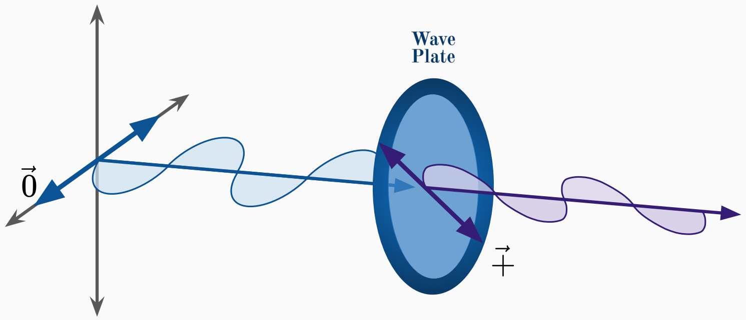


This is **unitary evolution**:

$$|\psi\rangle \longmapsto U|\psi\rangle$$

where $U$ is **unitary**: it preserves lengths, so unit vectors stay unit vectors. Every operation on an isolated system is of this form.

## The Hadamard gate

The single most important single-qubit unitary:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

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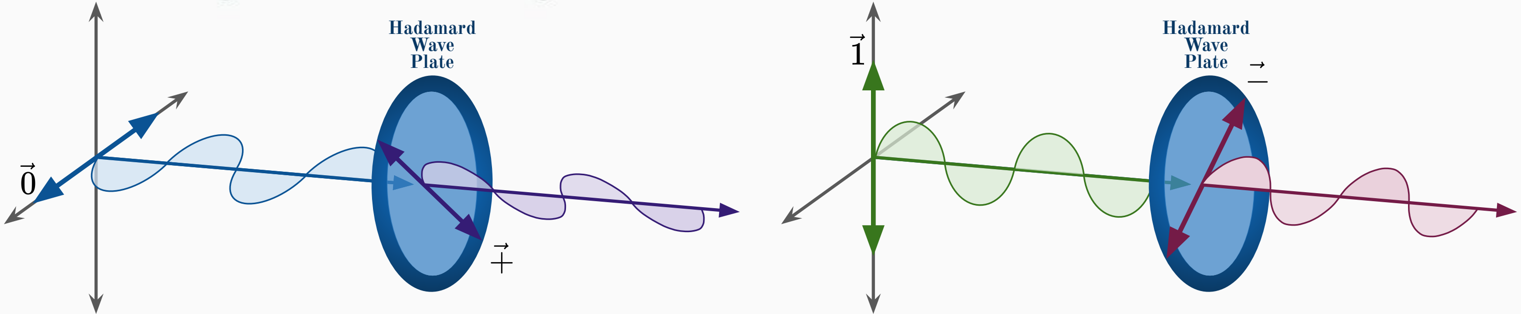
**Physically**: a wave plate that **mirrors the polarization about an axis at** $22.5°$ does exactly this — vertical light comes out diagonal.



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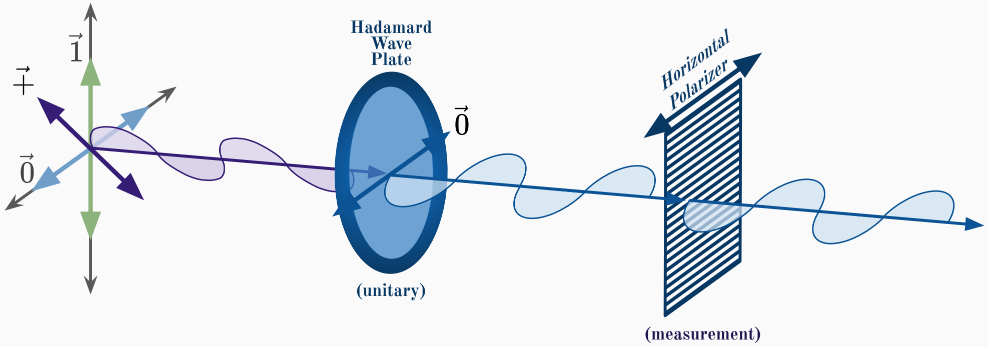
So here is a second way to tell $|+\rangle$ from $|-\rangle$: **apply $H$, then measure in the standard basis.**

Earlier we tilted the polarizer by 45°. Here we rotate the state instead and leave the measurement alone.

Same experiment, viewed two ways.



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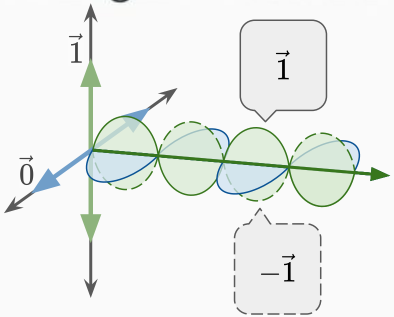

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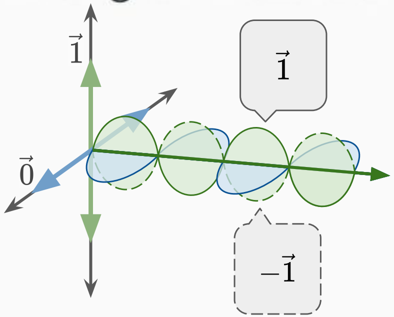


For any real $\theta$, the states

$$|\psi\rangle \qquad \text{and} \qquad e^{i\theta}\,|\psi\rangle$$

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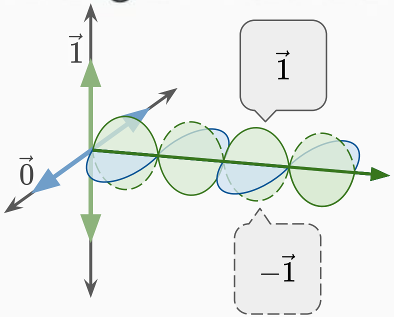


For any real $\theta$, the states

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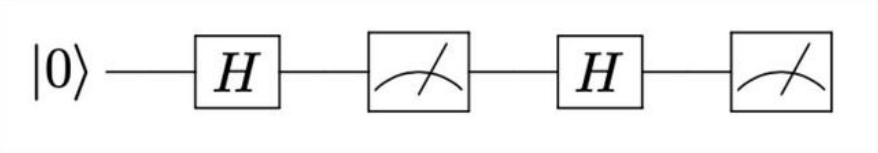
give the **same measurement statistics** after any unitary and any measurement.

No quantum process can detect a global phase.

# Experiment A




A wire carries one qubit; a box is a gate; the meter is a standard-basis measurement.

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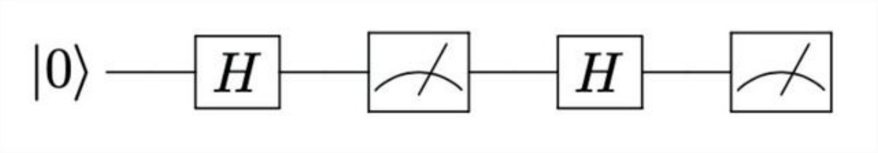


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**Outcome**: $|0\rangle$ and $|1\rangle$ with equal probability.

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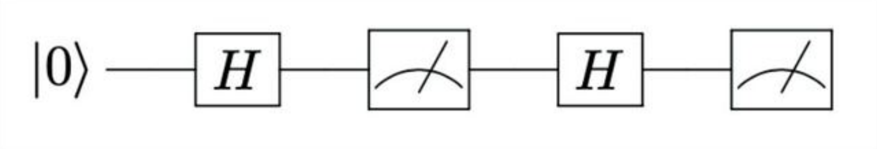
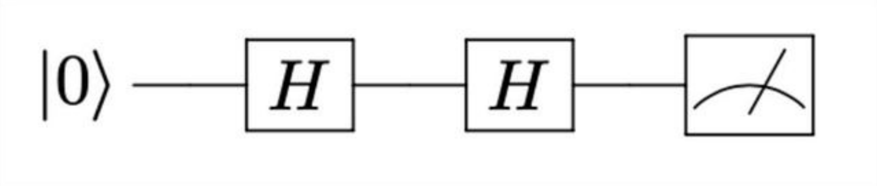



A wire carries one qubit; a box is a gate; the meter is a standard-basis measurement.

**Outcome**: $|0\rangle$ and $|1\rangle$ with equal probability.

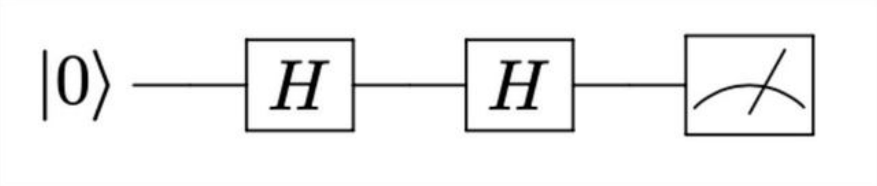
One might conclude that $H$ simply produces a random classical bit.

# Experiment B




The same circuit, but we measure **only at the end**.

# Experiment B




The same circuit, but we measure **only at the end**.

**Outcome**: $|0\rangle$, always.

# Experiment B

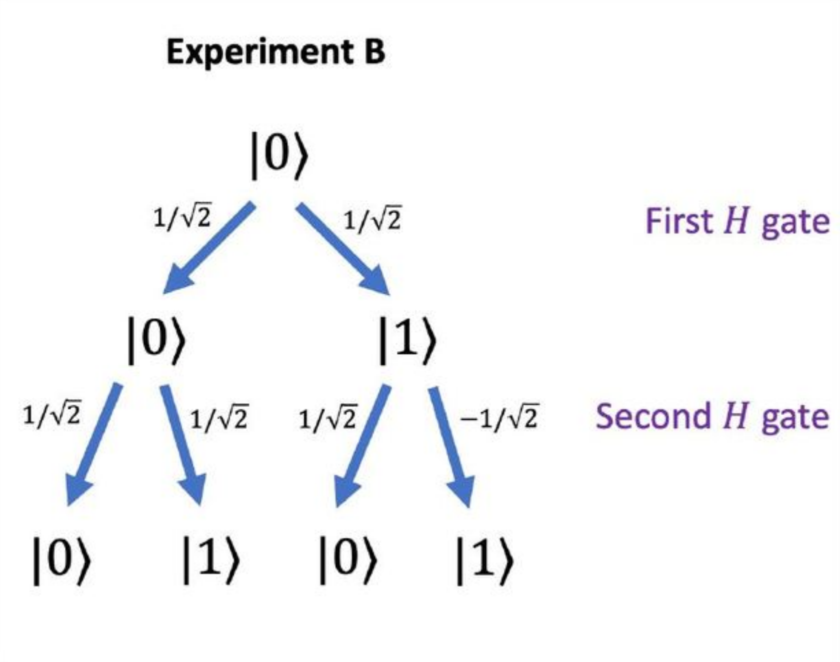
$$|0\rangle \;—\; \boxed{H} \;—\; \boxed{H} \;—\; \text{measure}$$

The same circuit, but we measure **only at the end**.

**Outcome**: $|0\rangle$, always.

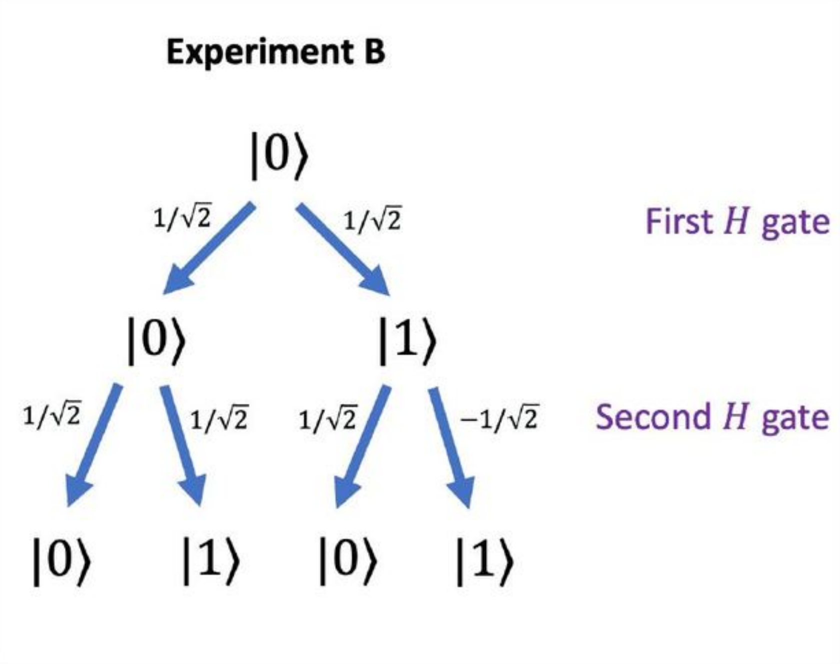
**Removing the intermediate measurement changes the result completely** — the circuit version of the three-polarizer surprise.

# Quantum interference




Applying several unitaries to a qubit gives a tree of "paths" between states.

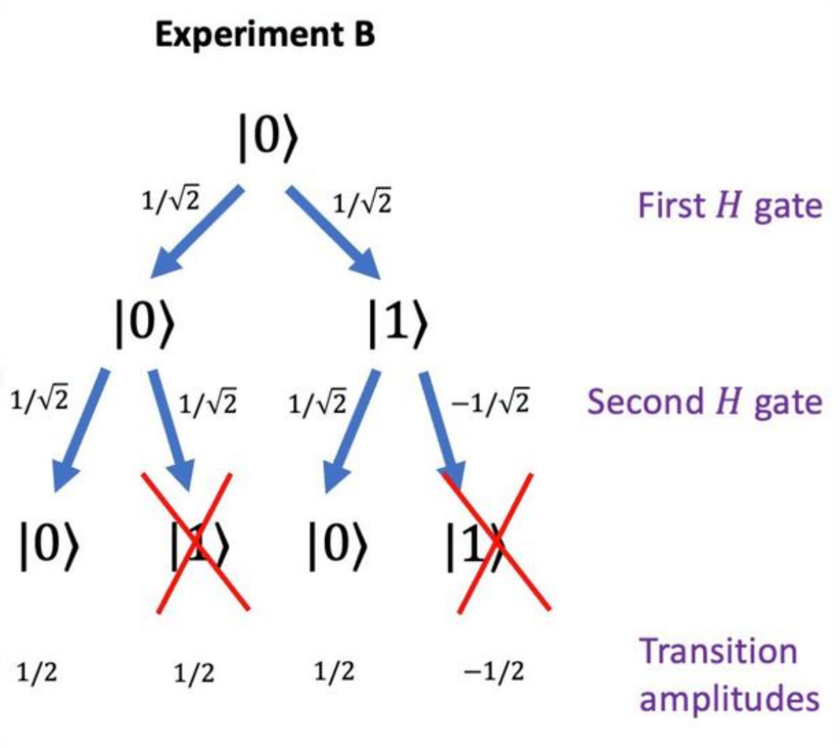
# Quantum interference




Applying several unitaries to a qubit gives a tree of "paths" between states.

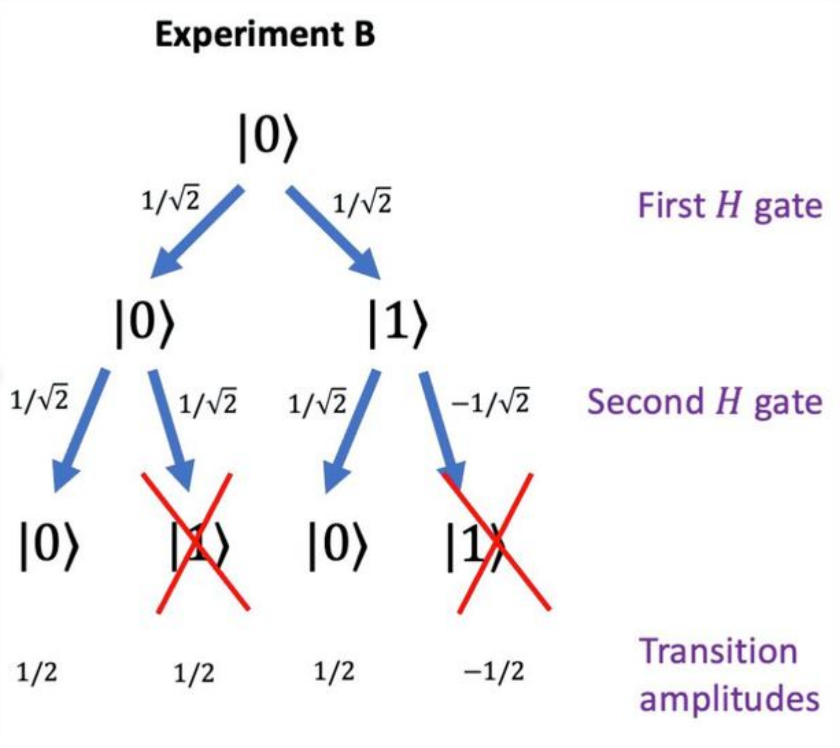
Each edge from $|a\rangle$ to $|b\rangle$ carries a **transition amplitude**, read off from the unitary.

# Quantum interference




A path's amplitude is the product of the amplitudes along it.

# Quantum interference

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Before measurement, the amplitude of ending at $|b\rangle$ is the **sum** over all paths ending at $|b\rangle$.

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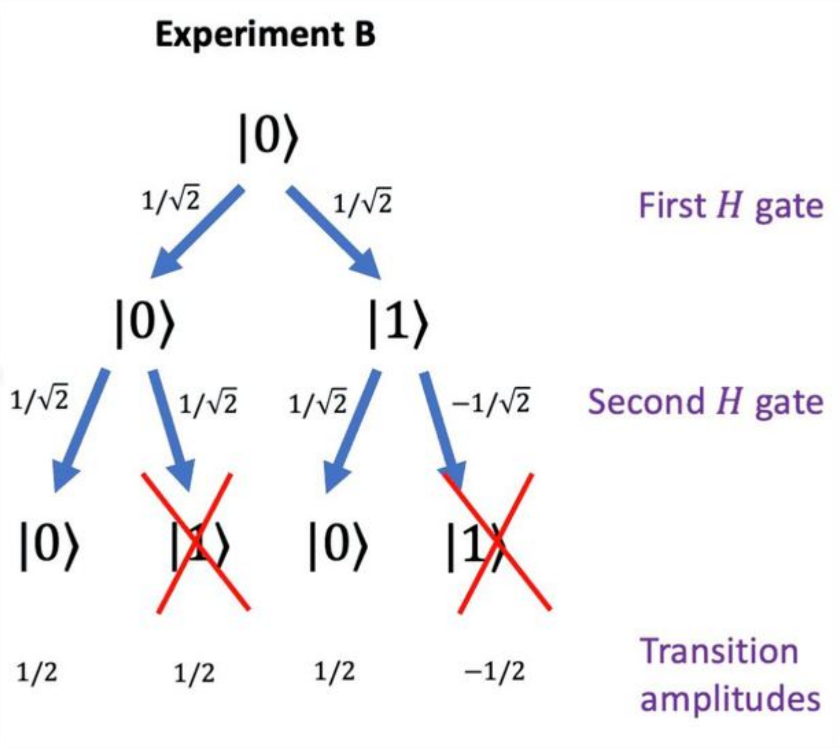
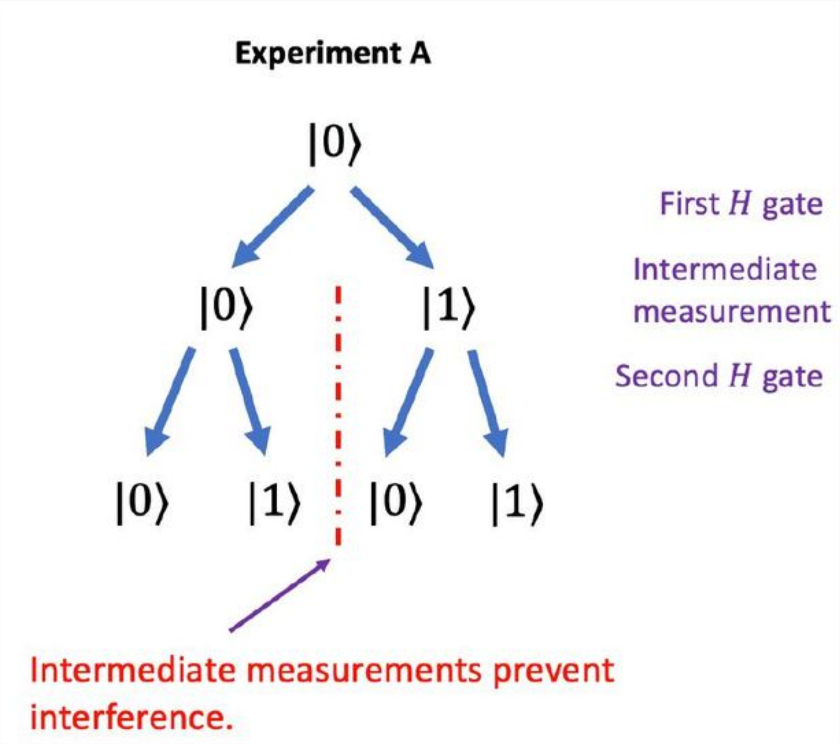



A path's amplitude is the product of the amplitudes along it.

Before measurement, the amplitude of ending at $|b\rangle$ is the **sum** over all paths ending at $|b\rangle$.

Amplitudes can be negative or complex, so paths can **constructively or destructively interfere**. In Experiment B, the two paths to $|1\rangle$ cancel.

# Quantum interference




With an intermediate measurement, only the paths consistent with the observed outcome are added up.

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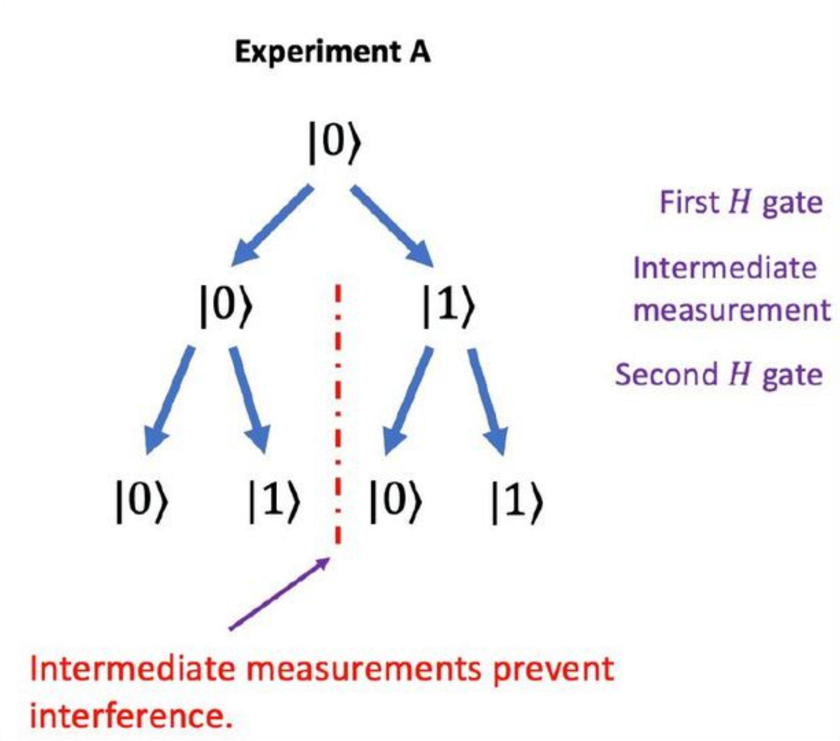


With an intermediate measurement, only the paths consistent with the observed outcome are added up.

**Takeaway**: measurement destroys superposition and prevents interference.

Barak Nehoran (Postdoc, CS Department, Columbia University)

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\rho = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\phi\rangle\langle\phi|$$

# Quantum Notation

*It's all just fancy vectors!*

$$|\Psi_{\text{EPR}}\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$$

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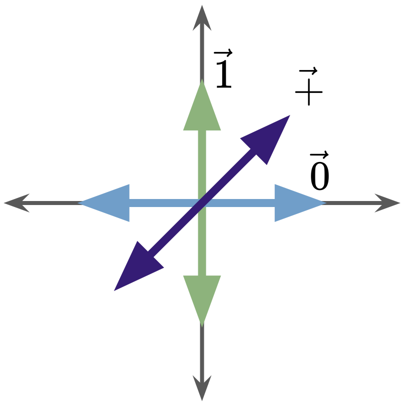


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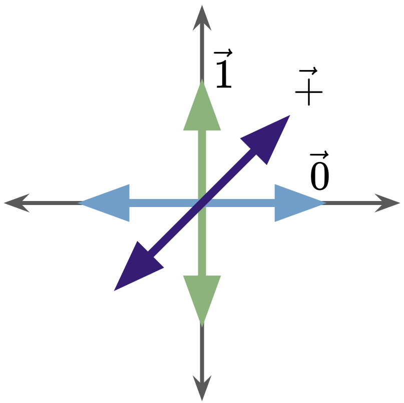


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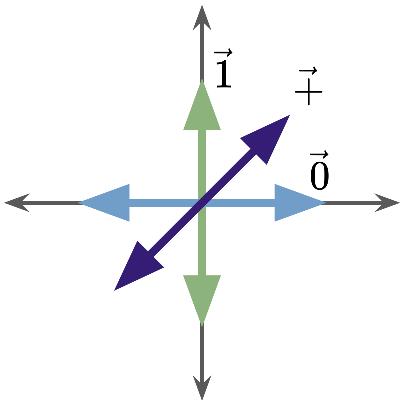


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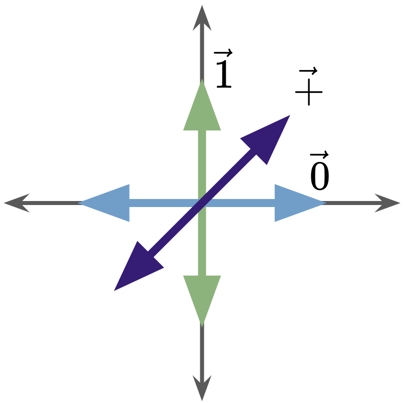


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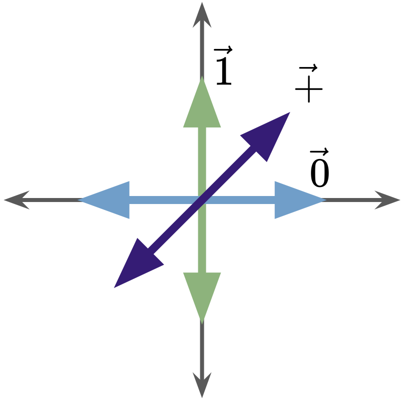


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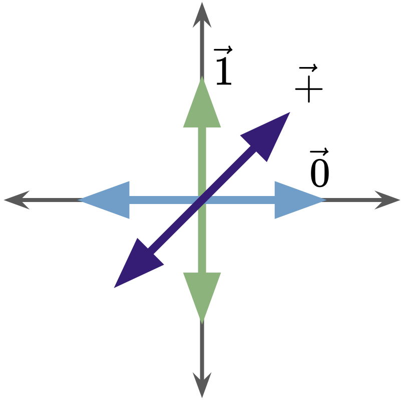


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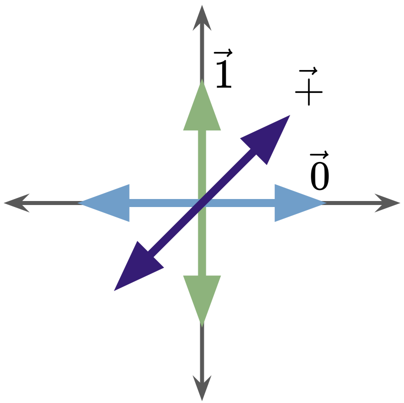


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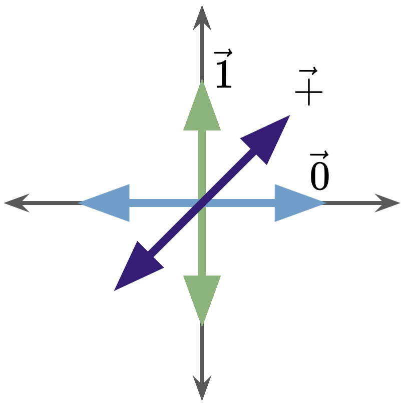


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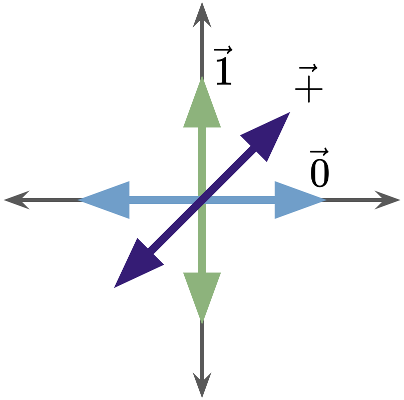


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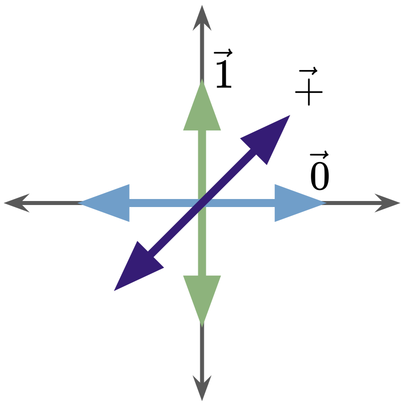


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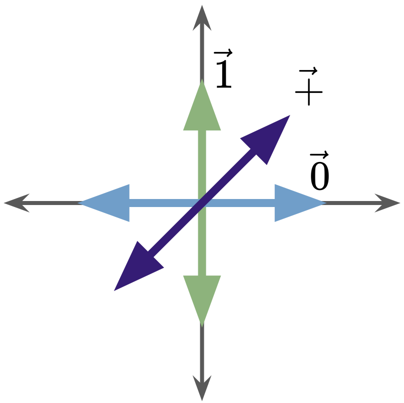


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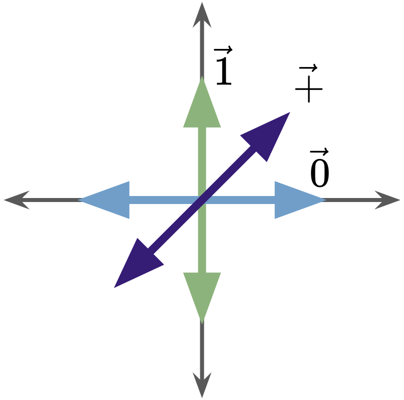


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"bracket"

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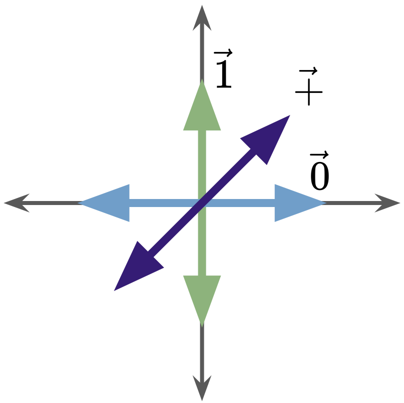


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“bracket”

“bra”

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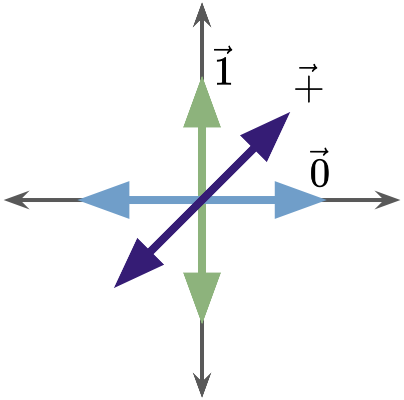


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"bracket" "bra" "ket"

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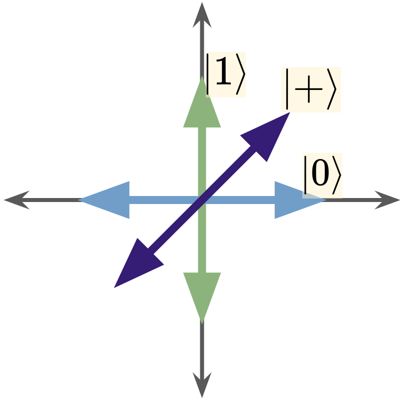


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“bracket”

$\begin{pmatrix} 0 & 1 \end{pmatrix} = \langle 1|$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

# Bras are conjugated rows

A ket is a column vector; its **dual** (Hermitian conjugate) is a row vector, the bra:

$$\langle 0| = \begin{pmatrix} 1, & 0 \end{pmatrix} \qquad \langle 1| = \begin{pmatrix} 0, & 1 \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \langle \psi| = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \end{pmatrix}$$

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$\overline{\alpha}, \overline{\beta}$ are the **complex conjugates** of $\alpha, \beta$ — which matters as soon as we allow complex amplitudes, as in circular polarization.

# Inner products

For $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and $|\theta\rangle = \gamma\,|0\rangle + \delta\,|1\rangle$:

$$\langle\psi \mid \theta\rangle = \begin{pmatrix}\overline{\alpha}, & \overline{\beta}\end{pmatrix}\begin{pmatrix}\gamma \\ \delta\end{pmatrix} = \overline{\alpha}\gamma + \overline{\beta}\delta.$$

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Two familiar facts in this language: $\langle \psi \mid \psi \rangle = 1$ is normalization, and the probability of outcome $|m\rangle$ is $|\langle m \mid \psi \rangle|^2$.

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For $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and $|\theta\rangle = \gamma\,|0\rangle + \delta\,|1\rangle$:

$$\langle \psi \mid \theta \rangle = \begin{pmatrix} \overline{\alpha}, & \overline{\beta} \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \overline{\alpha}\gamma + \overline{\beta}\delta.$$

Two familiar facts in this language: $\langle \psi \mid \psi \rangle = 1$ is normalization, and the probability of outcome $|m\rangle$ is $|\langle m \mid \psi \rangle|^2$.

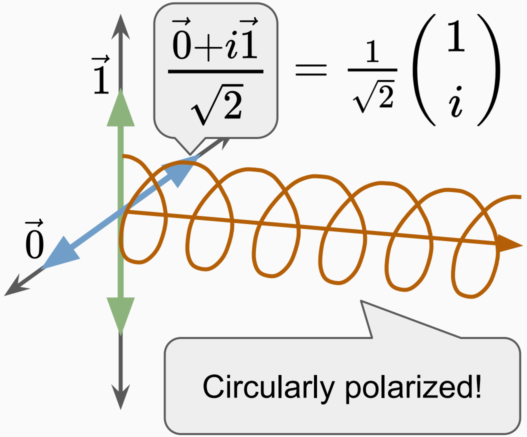
The order matters:

$$\langle \theta \mid \psi \rangle = \overline{\gamma}\alpha + \overline{\delta}\beta = \overline{\langle \psi \mid \theta \rangle}.$$

# A numerical example

Circularly polarized light:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}.$$



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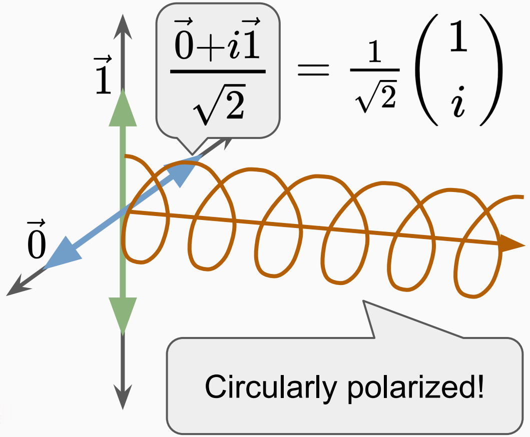


Its bra is

$$\langle\psi| = \frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}.$$

**Note the conjugate:** $\bar{i} = -i$.

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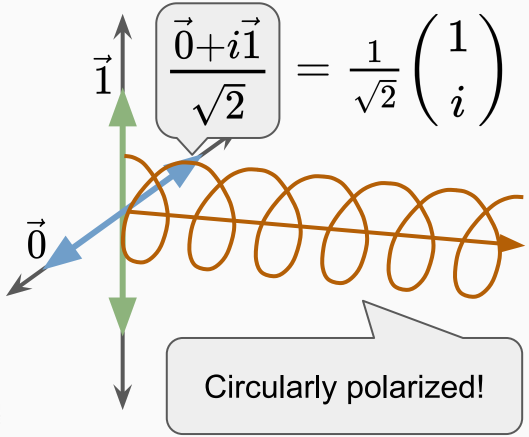


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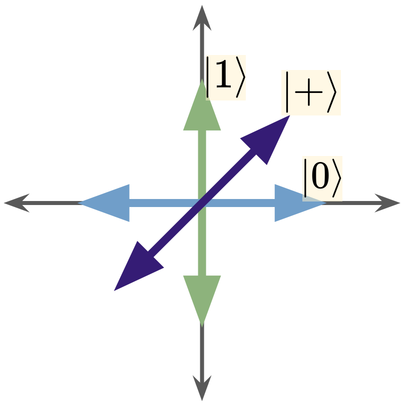
Check normalization:

$$\langle\psi \mid \psi\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1+(-i)i}{2} = 1.$$

Barak Nehoran (Postdoc, CS Department, Columbia University)

# Quantum Notation: *It's all just fancy vectors!*

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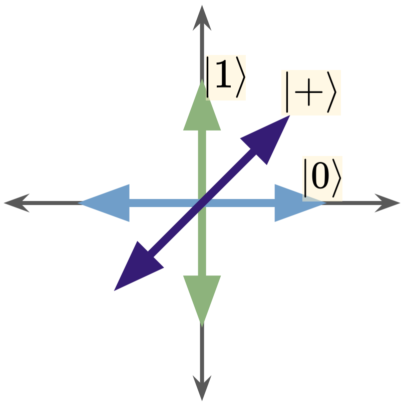


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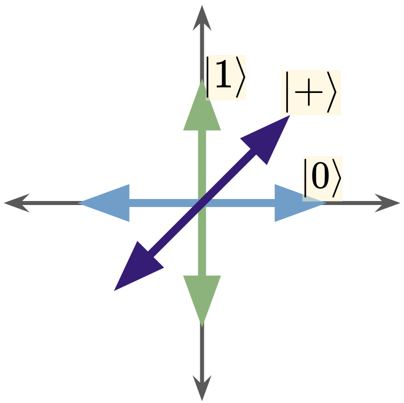


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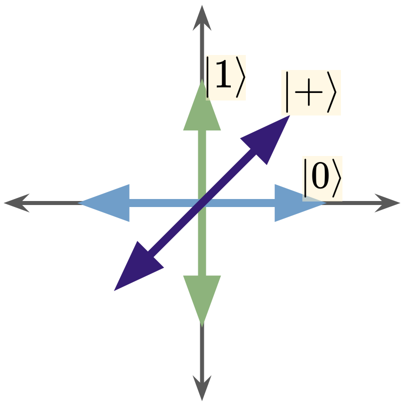


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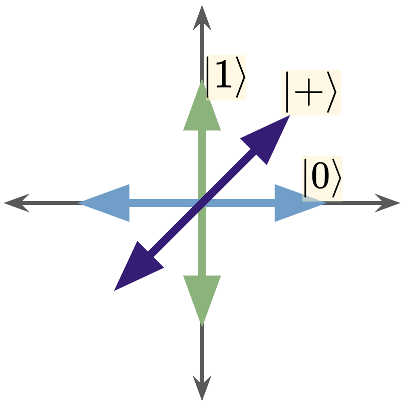


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distribute

$$\Pr\left[\begin{array}{l}\text{we get a } |1\rangle \text{ if we} \\ \text{measure the state } |+\rangle\end{array}\right] = \left|\langle 1|+\rangle\right|^2 = \left|\langle 1|\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\right|^2$$

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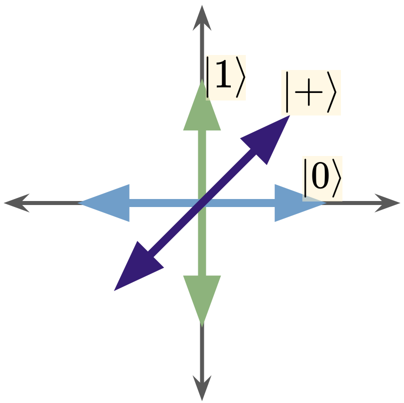


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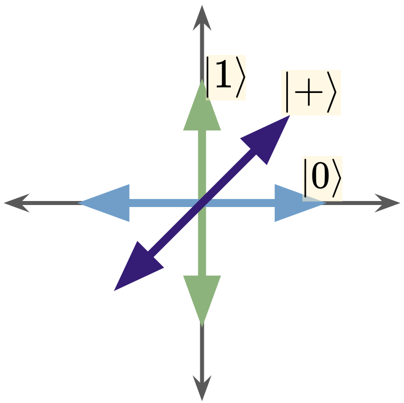


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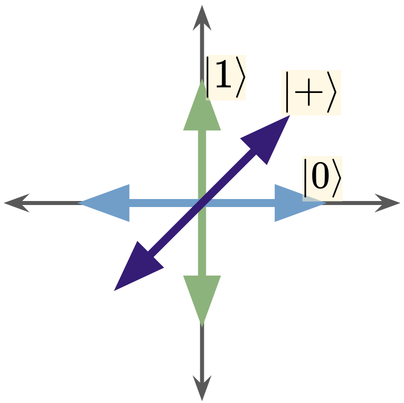


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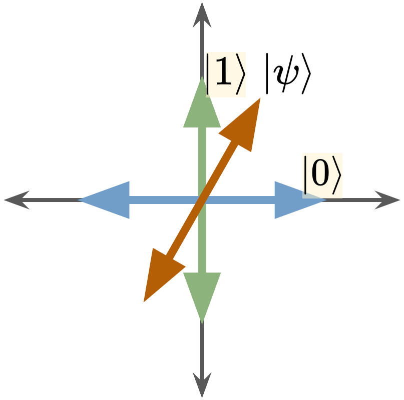


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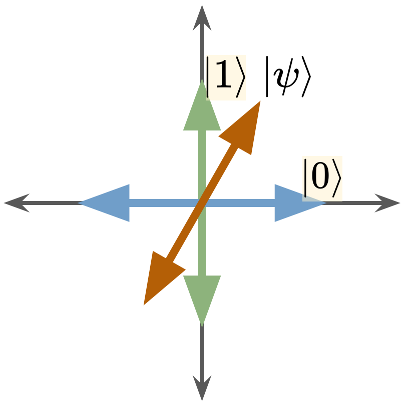


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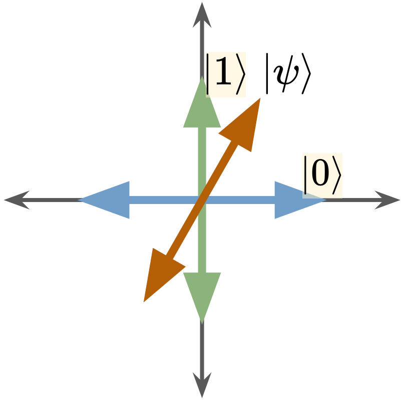


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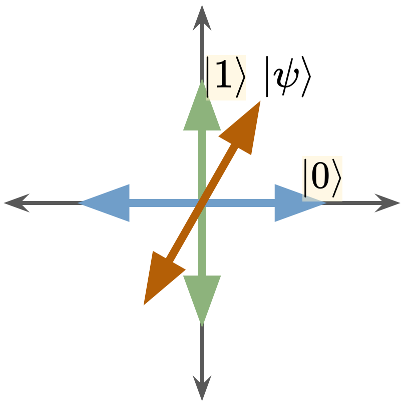


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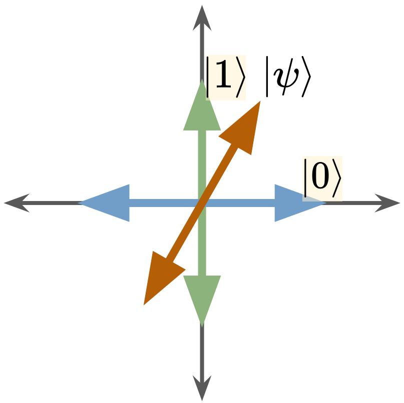


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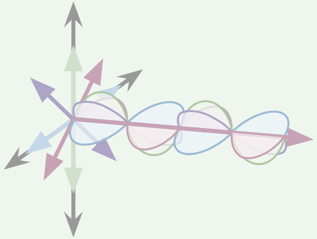
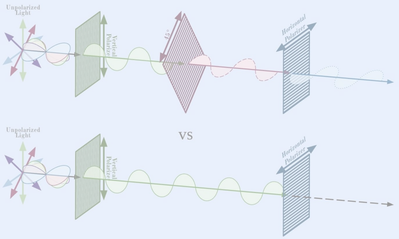
# Outline

1 The Qubit



2 Quantum Measurement

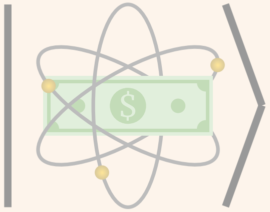


3 Quantum Entanglement & Many Qubits

First qubit Second qubit

$$\frac{1}{\sqrt{2}}|0\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|0\rangle$$

4 No Cloning and Quantum Money



# Two photons instead of one

If the first photon is in state $|\psi_1\rangle$ and the second in state $|\psi_2\rangle$, the joint state is written

$$|\psi_1\rangle \otimes |\psi_2\rangle\,, \qquad \text{or just} \qquad |\psi_1\rangle\,|\psi_2\rangle\,.$$

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Two qubits have **four** basis states:

$$|0\rangle\,|0\rangle \qquad |0\rangle\,|1\rangle \qquad |1\rangle\,|0\rangle \qquad |1\rangle\,|1\rangle\,,$$

often abbreviated $|0,0\rangle\,, |0,1\rangle\,, |1,0\rangle\,, |1,1\rangle$ or $|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle$.

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So a two-qubit state is a unit vector in the four-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $n$ qubits live in $2^n$ dimensions.

# Multiplying out the tensor product

If $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$, then

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha\gamma|0,0\rangle + \alpha\delta|0,1\rangle + \beta\gamma|1,0\rangle + \beta\delta|1,1\rangle\,.$$

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These four numbers are exactly the amplitudes $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}$ of a general two-qubit state:

$$|\psi\rangle = \sum_{i,j} \alpha_{ij}|i,j\rangle\,, \qquad \sum_{i,j} |\alpha_{ij}|^2 = 1.$$

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# Multiple Qubits

General two-qubit state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

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# Multiple Qubits

That is, this is any unit vector on these four basis vectors.

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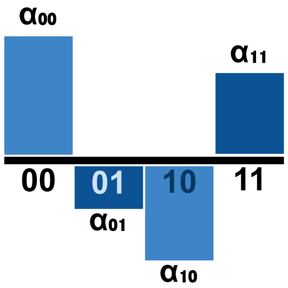

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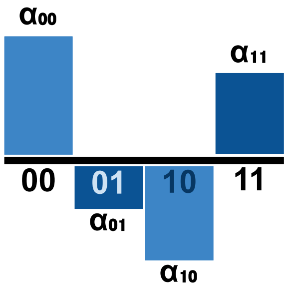
$|\psi\rangle =$ 


If we measure, the probability of getting outcome $x$ is $|\alpha_x|^2$

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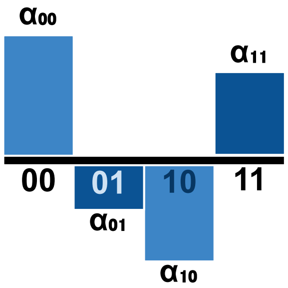
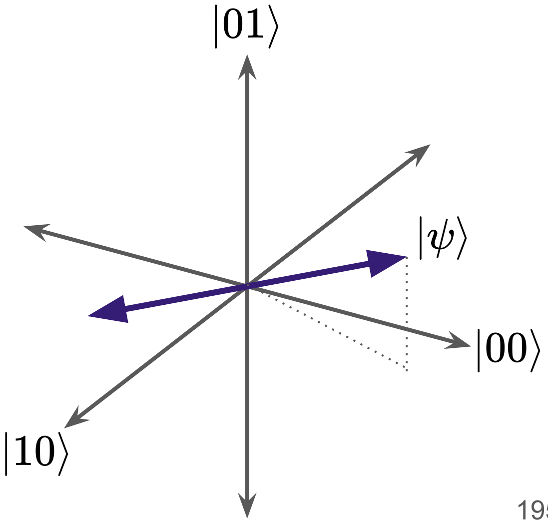
$|\psi\rangle =$ 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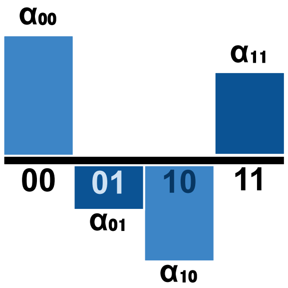
$|\psi\rangle =$ 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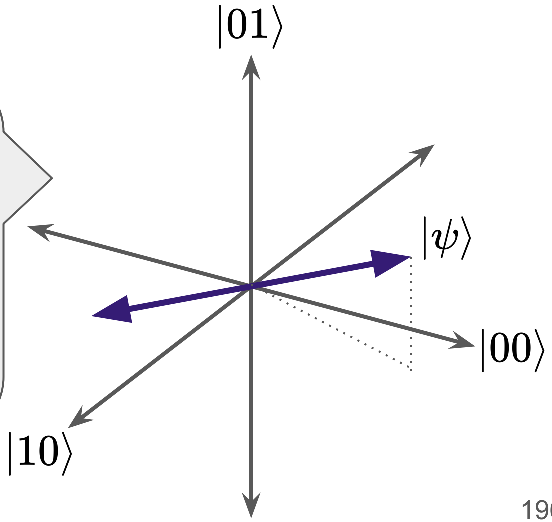
Unfortunately, we can't easily depict 4-dimensional vectors, so you'll have to imagine it! This is a 3D depiction.



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That is, this is any unit vector on these four basis vectors.

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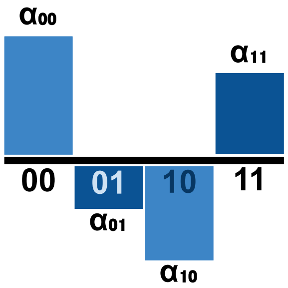


If we measure, the probability of getting outcome $x$ is $|\alpha_x|^2$

### Partial Measurement

If we measure **just the first qubit**,

We get outcome **0** with probability $|\alpha_{00}|^2 + |\alpha_{01}|^2$,

and the remaining quantum state is the state

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{|\alpha_{00}|^2 + |\alpha_{01}|^2}$$

the normalized projection onto the subspace spanned by $|00\rangle$ and $|01\rangle$.

# Examples of two-qubit states

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$$\begin{aligned}
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&= \frac{1}{2}\Big( |0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle \Big)
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The first two were built from single-qubit states. **Can the last two be?**

# Product states and entangled states

A state that can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$ is called a **product state**.

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Try it for $\frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$. We would need

$$\left(a\,|0\rangle + b\,|1\rangle\right) \otimes \left(c\,|0\rangle + d\,|1\rangle\right) = ac\,|0,0\rangle + ad\,|0,1\rangle + bc\,|1,0\rangle + bd\,|1,1\rangle\,,$$

so $ad = \frac{1}{\sqrt{2}}$ and $bc = -\frac{1}{\sqrt{2}}$, while $ac = bd = 0$.

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A state that is not a product state is **entangled**. This one and $\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$ are both called **EPR pairs** or **Bell pairs**.

Barak Nehoran (Postdoc, CS Department, Columbia University)

# Quantum Entanglement

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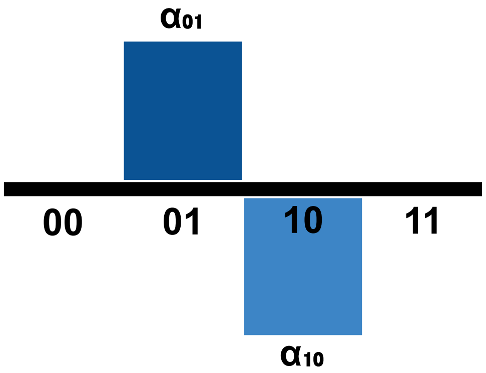

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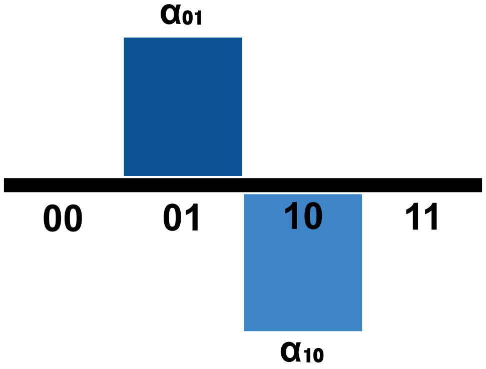

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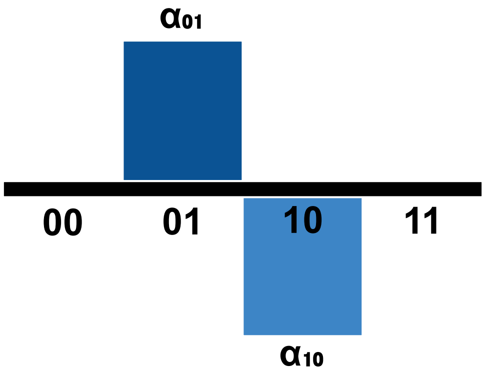

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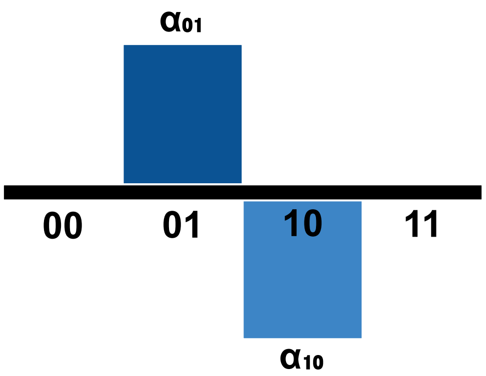


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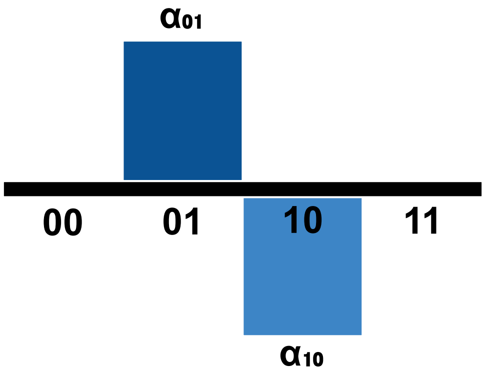


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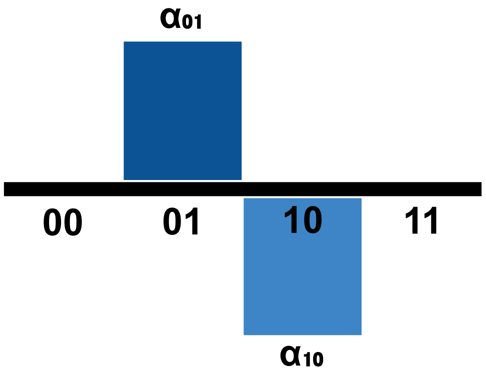


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**First qubit** **Second qubit**

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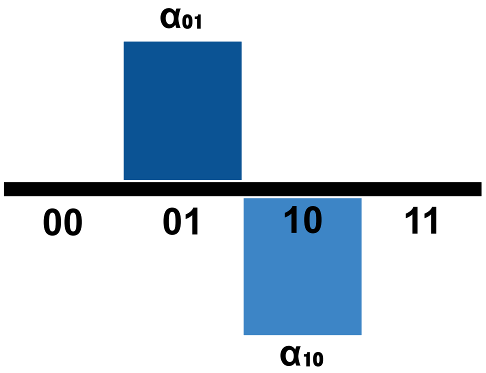


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$$\alpha_{00} = \frac{1}{\sqrt{2}}$$

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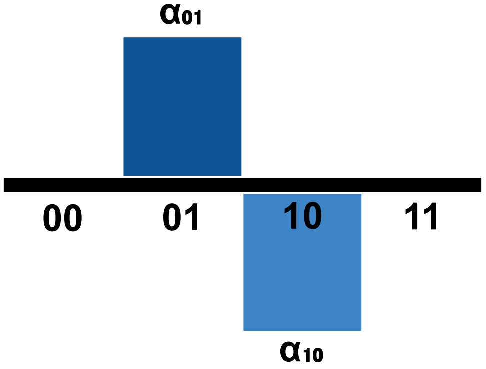


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$$\alpha_{00} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad |\alpha_{00}|^2 = \frac{1}{2}$$

?

# Quantum Entanglement

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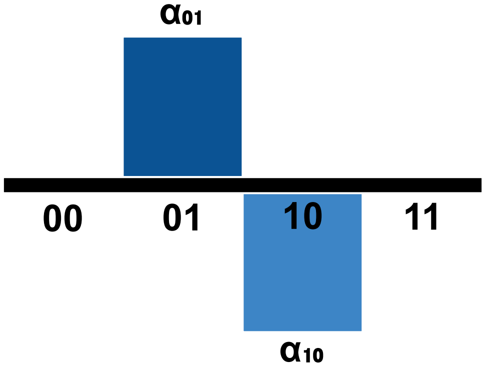


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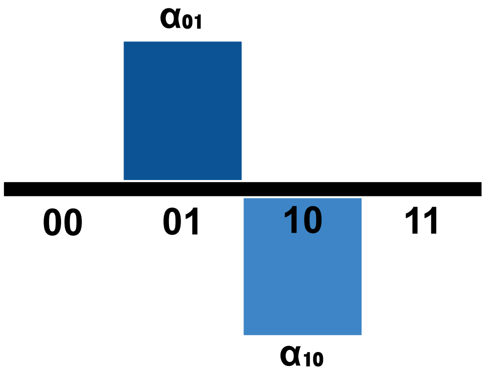


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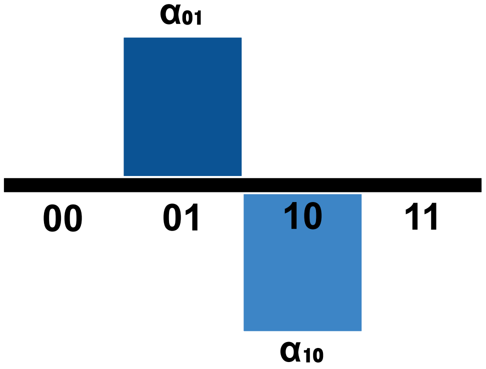


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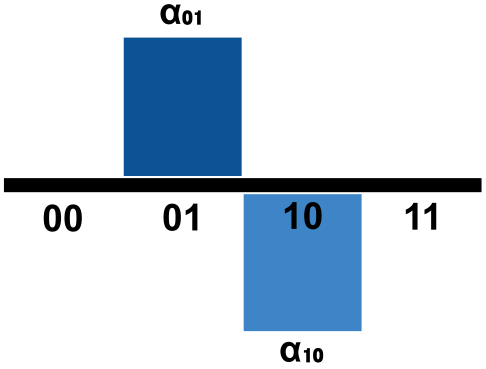


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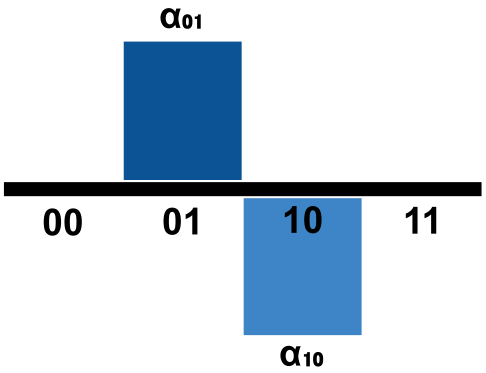


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**Fact**: This is the case in any basis!

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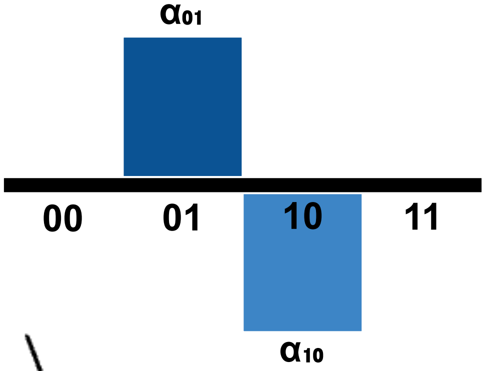
$$= \frac{1}{\sqrt{2}}|+\rangle|-\rangle - \frac{1}{\sqrt{2}}|-\rangle|+\rangle$$

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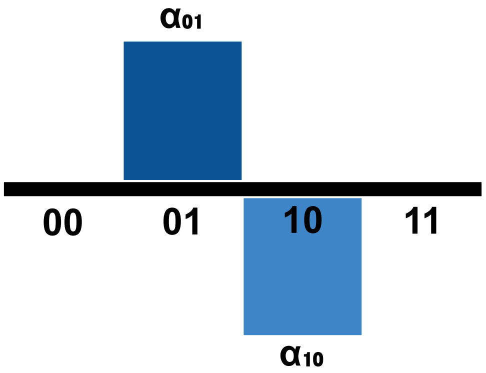


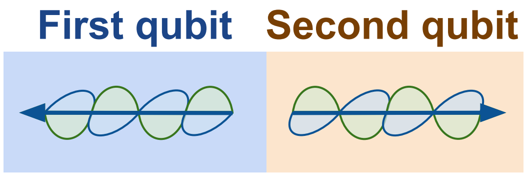

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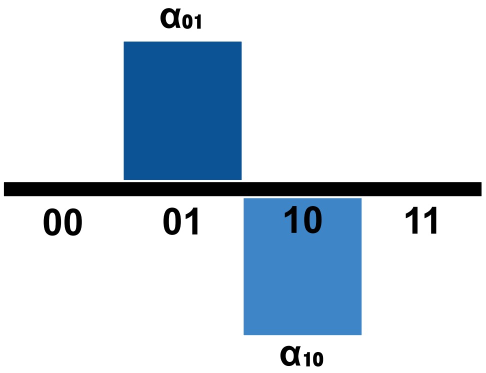


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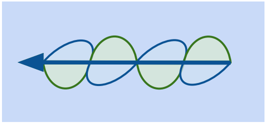
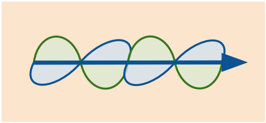


Second qubit



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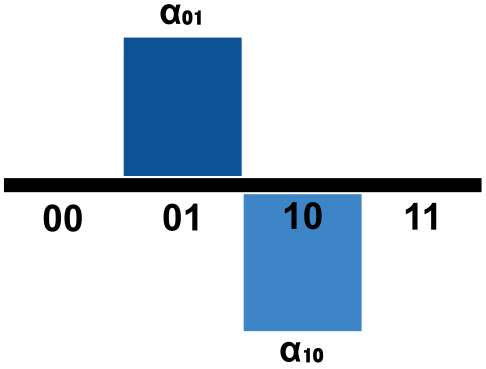


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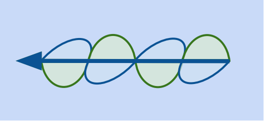
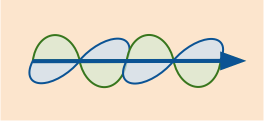


Second qubit



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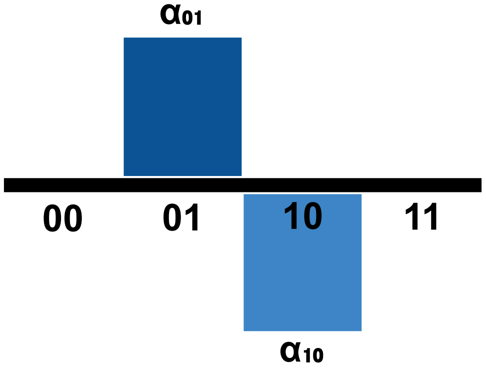


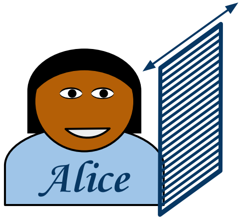


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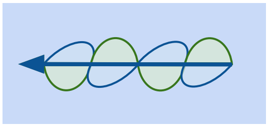
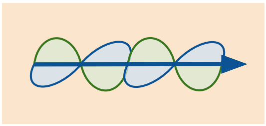
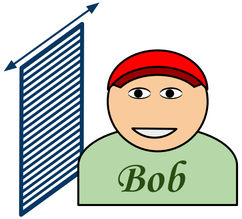


Second qubit





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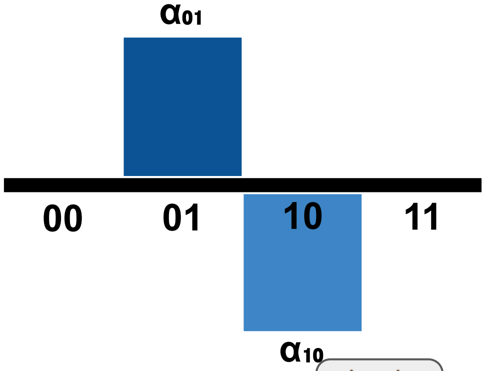


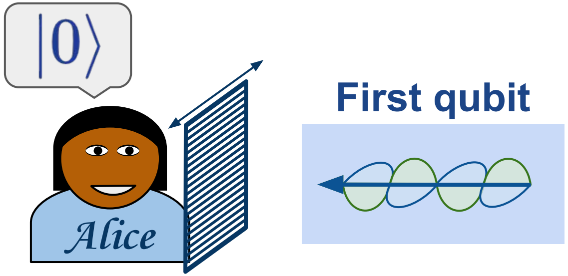


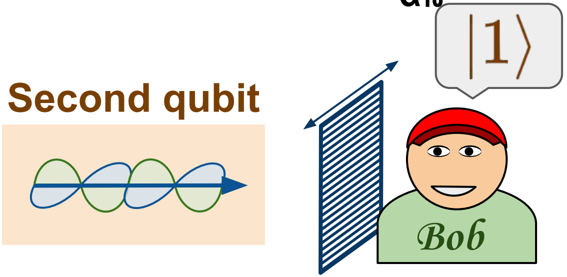

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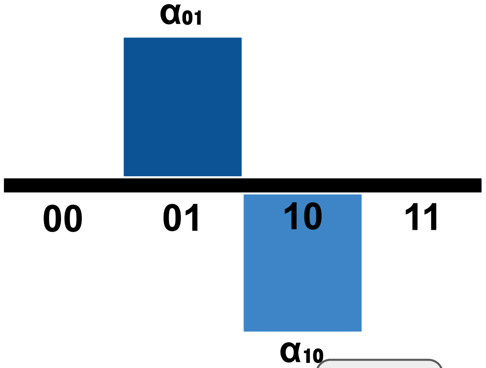


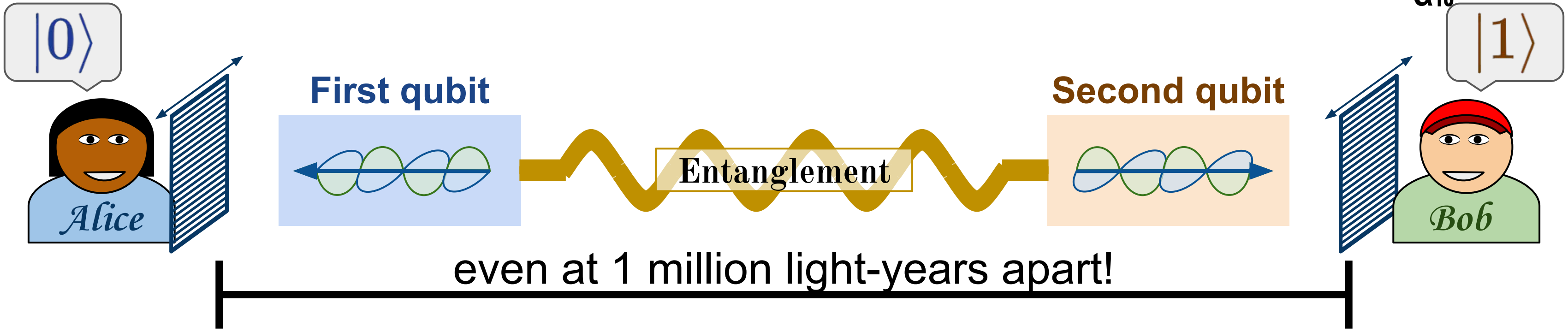

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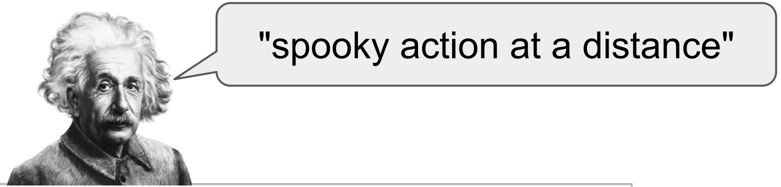


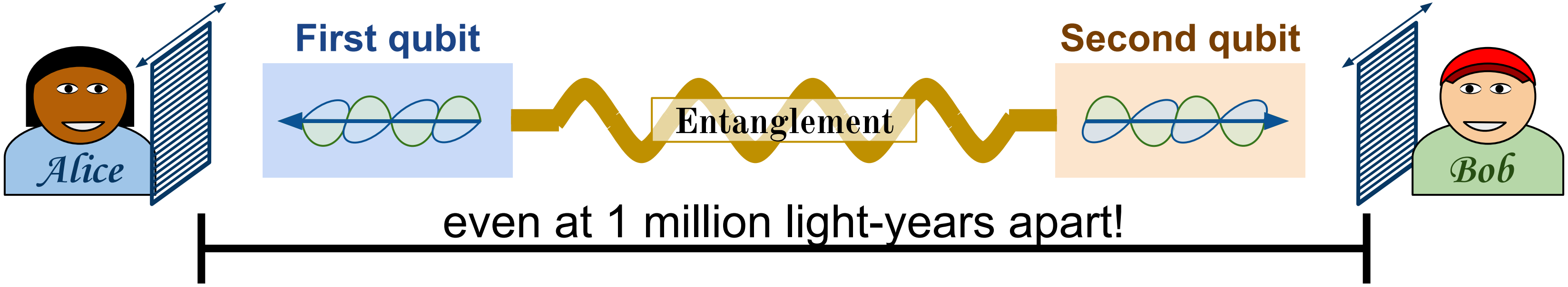

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

## Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function



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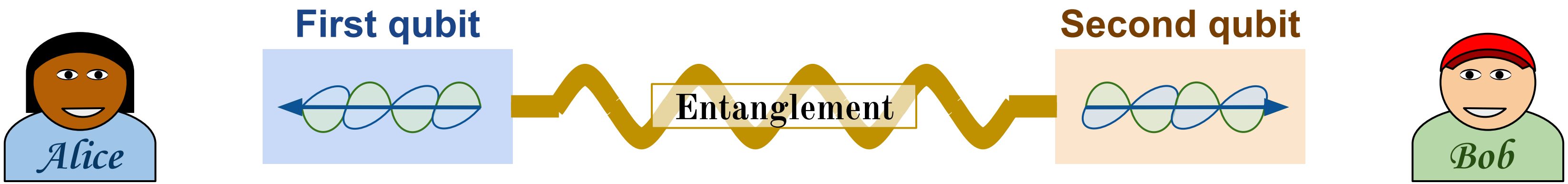
# The No-Communication Theorem



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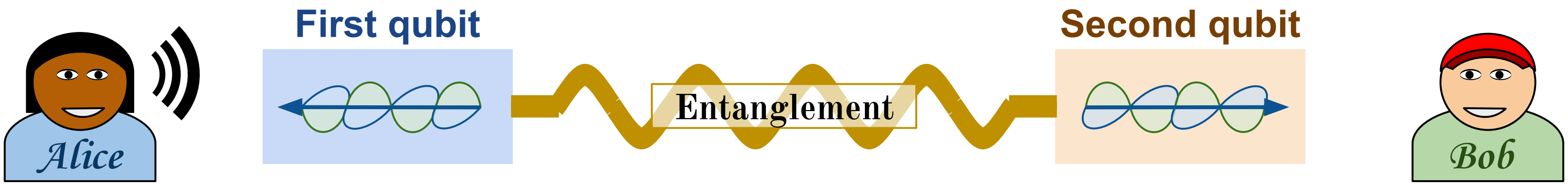

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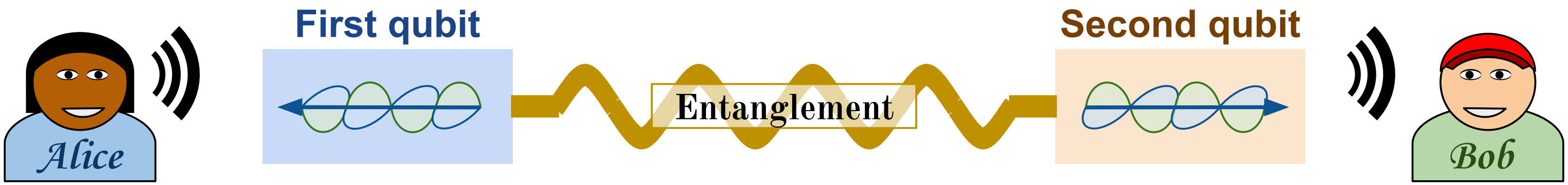
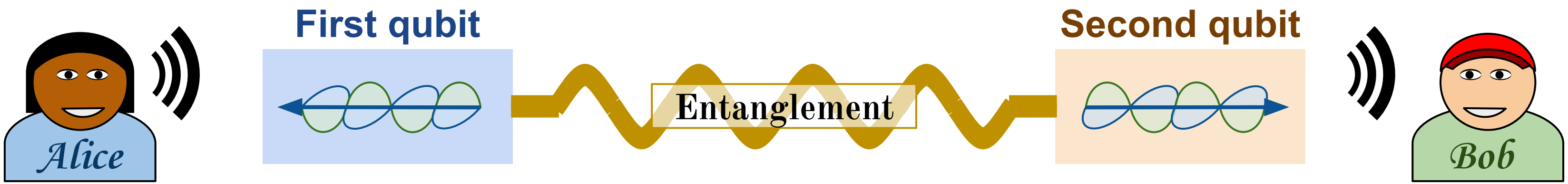


# The No-Communication Theorem

**Theorem**: Alice and Bob **cannot** use their entangled qubits to communicate information.



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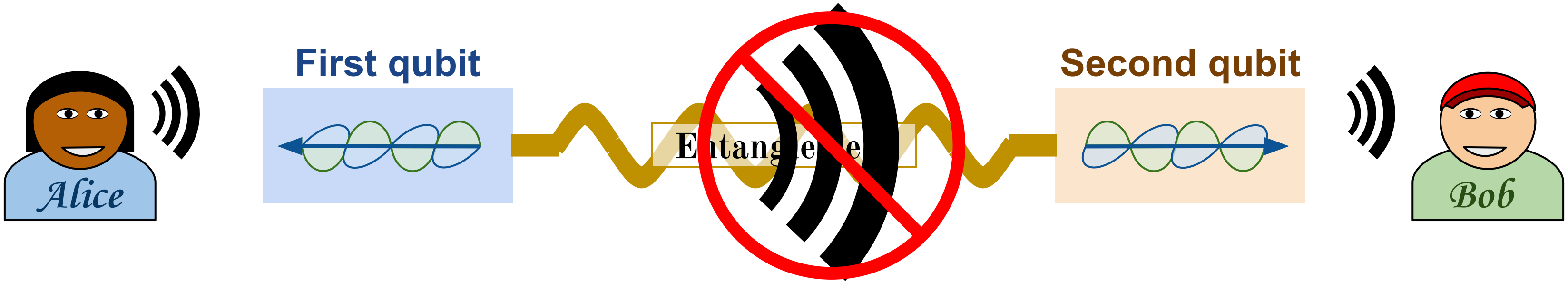
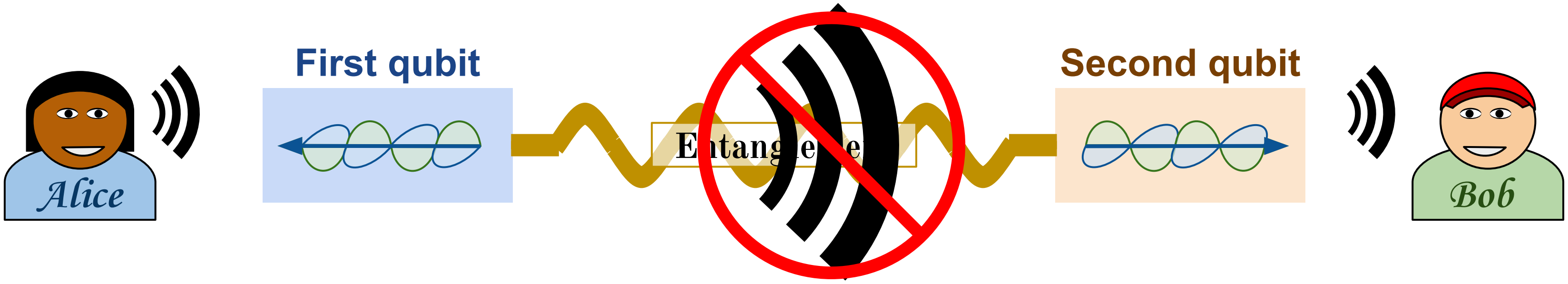


Barak Nehoran (Postdoc, CS Department, Columbia University)

# The No-Communication Theorem

**Theorem**: Alice and Bob **cannot** use their entangled qubits to communicate information.

That is, they can get **classically impossible correlations**, but **no communication**.



# Unitaries on two qubits

If $U, V$ are single-qubit unitaries, then $U \otimes V$ acts on each qubit separately:

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But just as not every state is a product state, **not every two-qubit unitary is a product unitary**.

Those are the interesting ones: they can create entanglement out of unentangled inputs.

# CNOT makes a Bell pair

**CNOT** flips the second qubit exactly when the first is $|1\rangle$:

$$\begin{aligned} |0,0\rangle &\mapsto |0,0\rangle\,, & |0,1\rangle &\mapsto |0,1\rangle\,, \\ |1,0\rangle &\mapsto |1,1\rangle\,, & |1,1\rangle &\mapsto |1,0\rangle\,. \end{aligned}$$

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Starting from two photons, both vertical:

$$|0,0\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}\Big(\,|0,0\rangle + |1,0\rangle\,\Big) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}\Big(\,|0,0\rangle + |1,1\rangle\,\Big).$$

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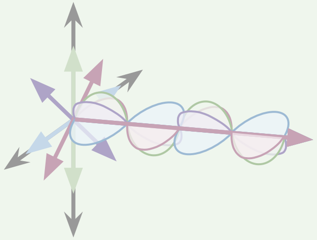
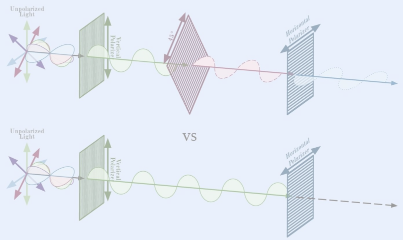
**A two-gate circuit produces the entangled pair Alice and Bob were sharing.**

# Outline

1 The Qubit



2 Quantum Measurement

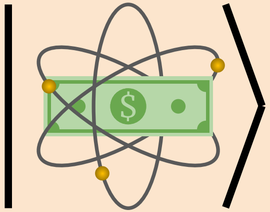


3 Quantum Entanglement & Many Qubits

First qubit Second qubit

$$\frac{1}{\sqrt{2}}|0\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|0\rangle$$

4 No Cloning and Quantum Money



Barak Nehoran (Postdoc, CS Department, Columbia University)

# A Brief History of Quantum Cryptography

Public-key Cryptography

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**Quantum Money**
[Stephen Wiesner]



1976
Diffie Hellman Key Exchange
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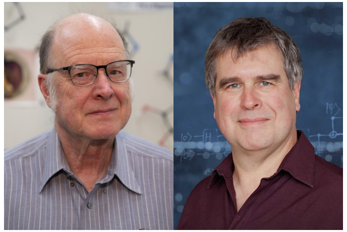

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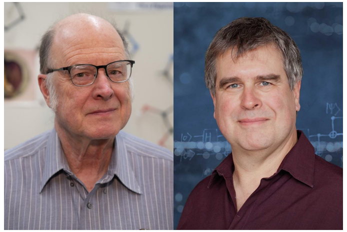

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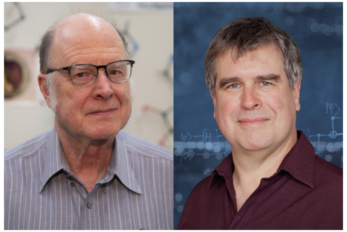

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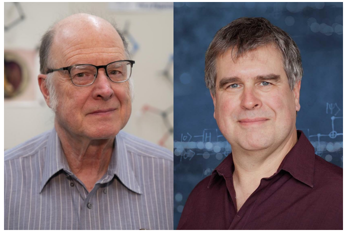

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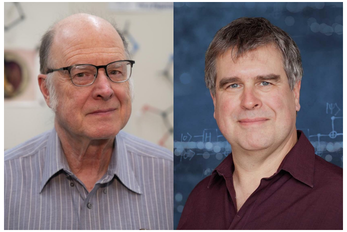
**While he was a PhD student at Columbia University**



# Cloning and the No-cloning Theorem




**Cloning** = Making a faithful copy of a quantum state

The **No-Cloning Theorem** [Park 1970, Dieks 1982, Wootters & Zurek 1982]:

An *unknown* quantum state **cannot be cloned**.

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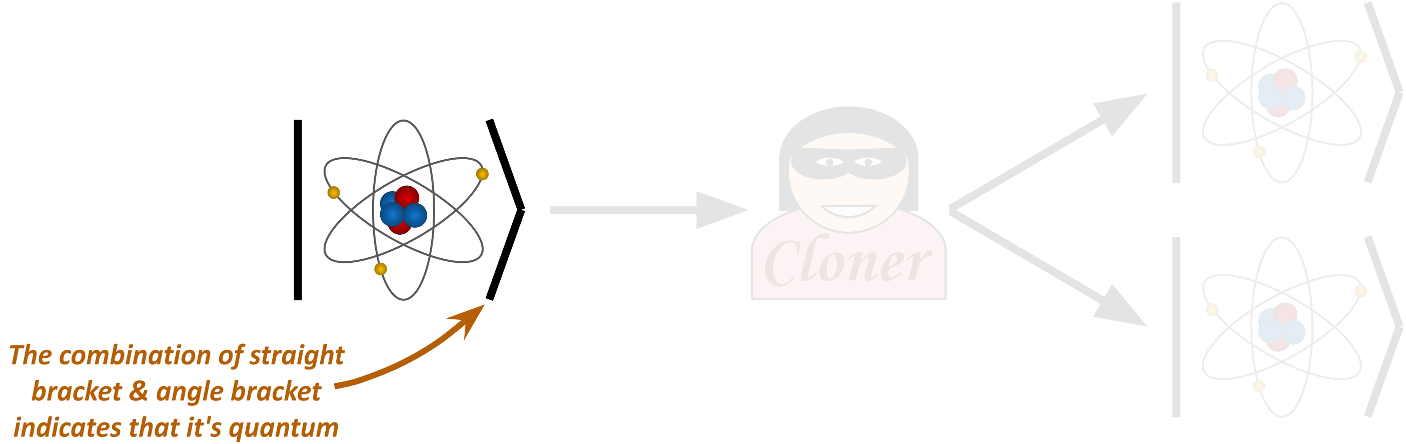


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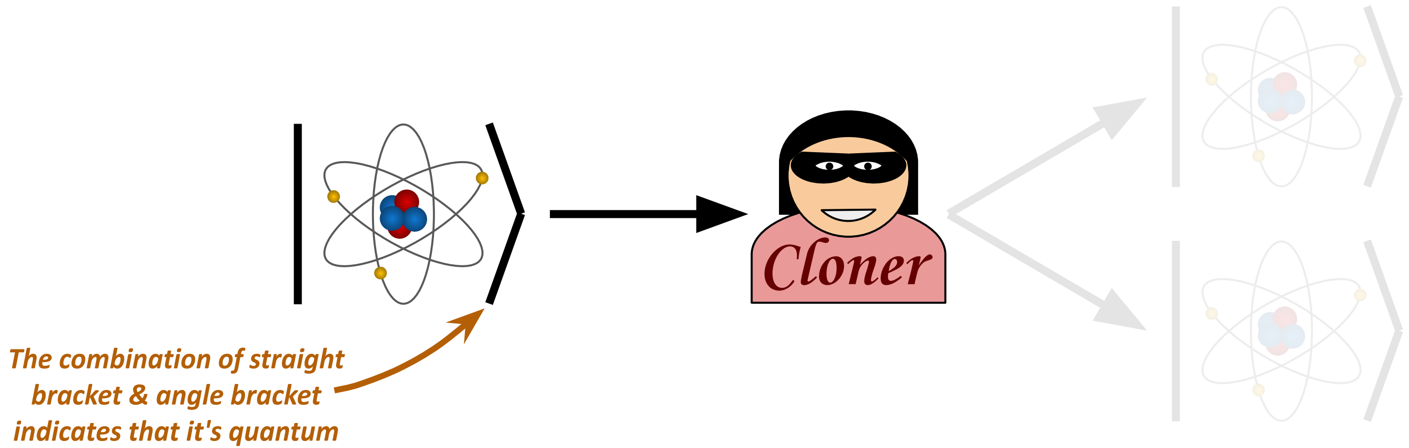


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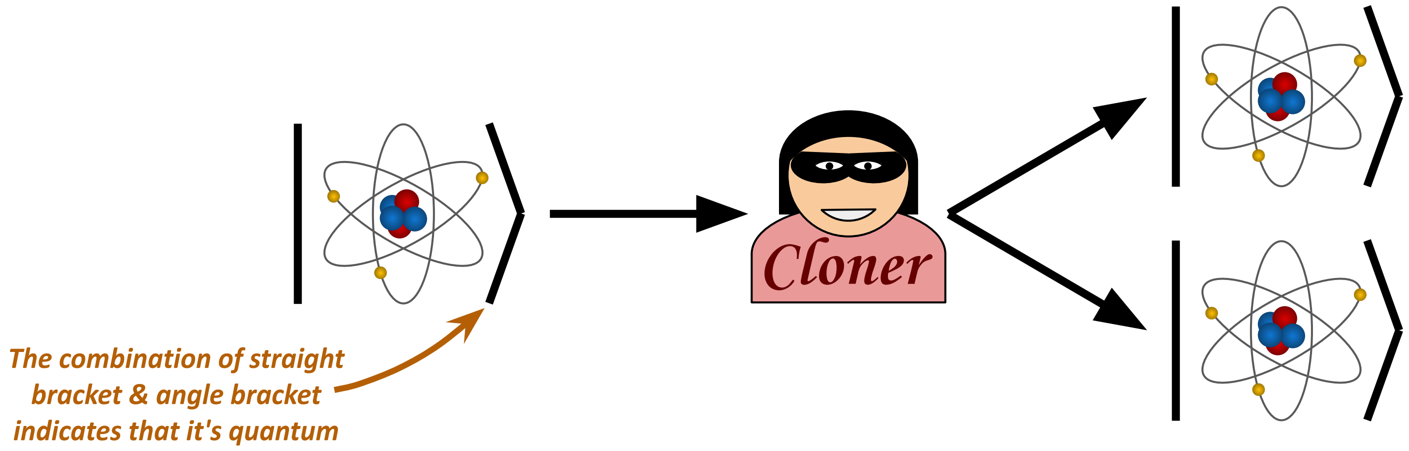


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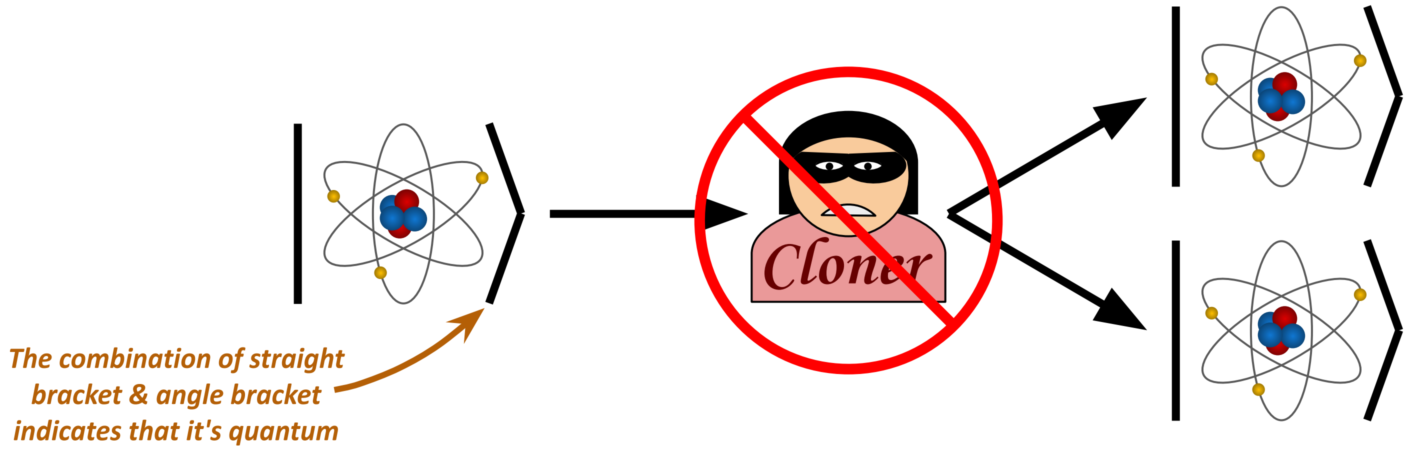


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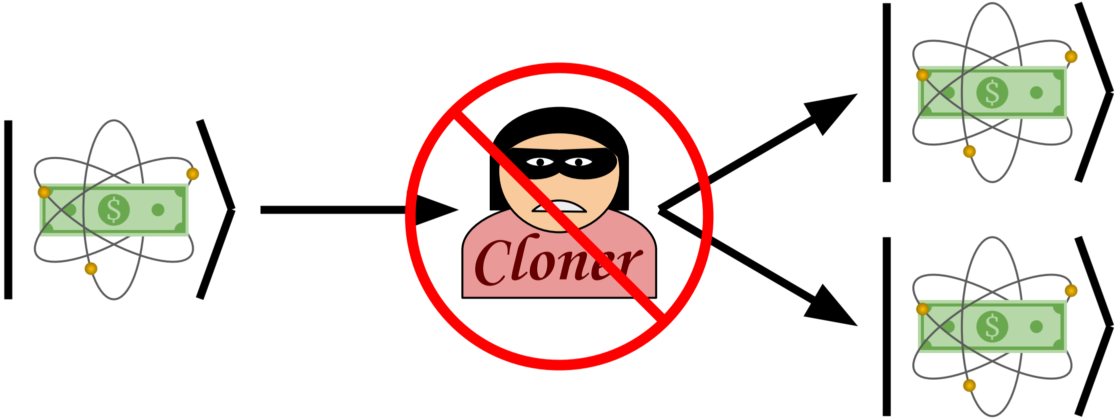


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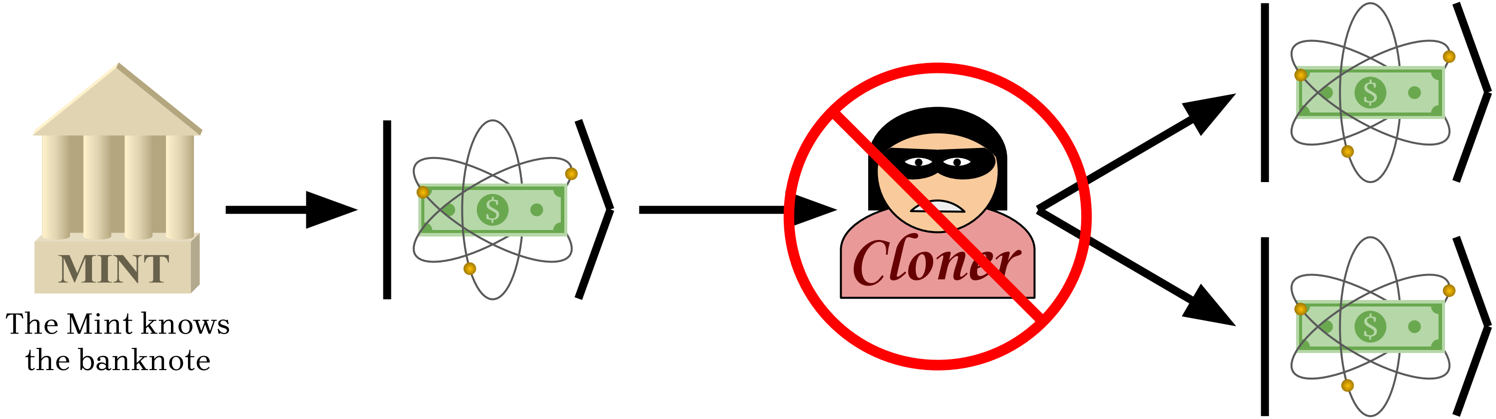


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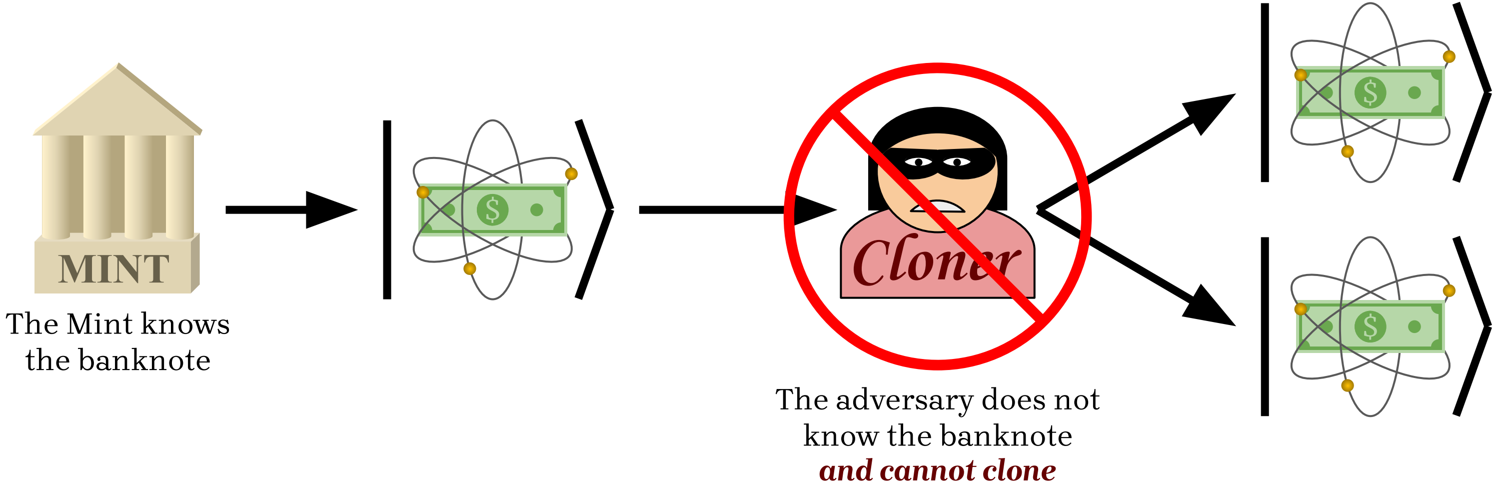


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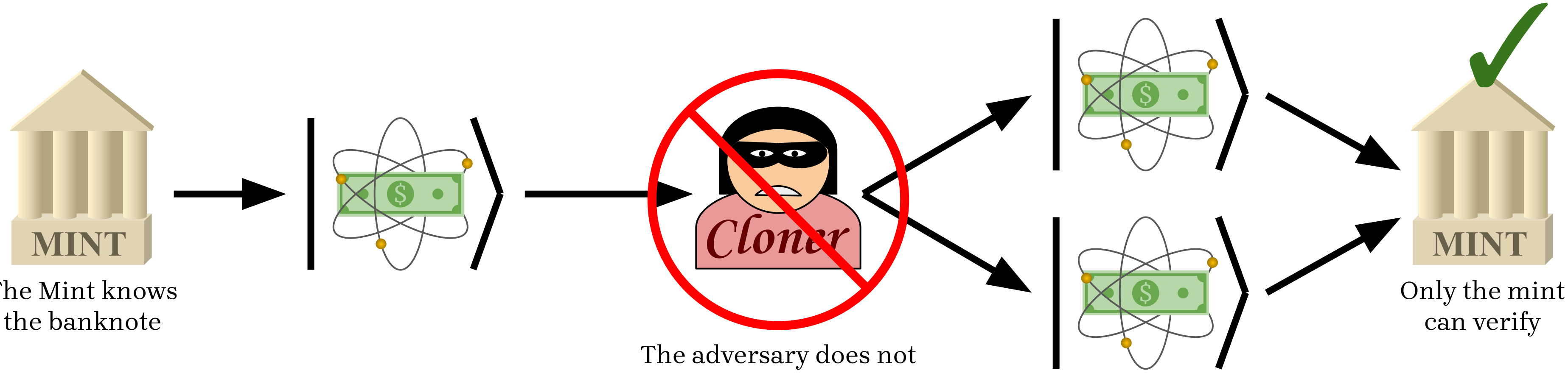


The Mint knows the banknote

The adversary does not know the banknote *and cannot clone*

Only the mint can verify

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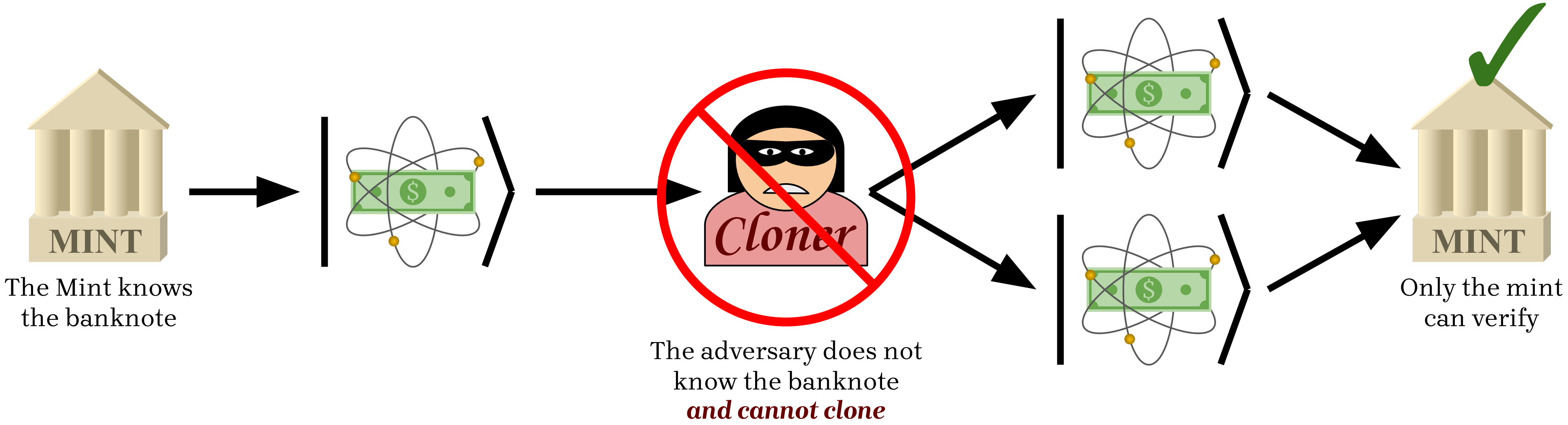


**Quantum money was the *first* quantum cryptographic protocol**
**(and was the inspiration for Quantum Key Distribution [BB84])**

| Physical Currency | Digital Currency |
|---|---|
|  |  |
| Coins, cash, gold, shells, valuable goods | Credit cards, checks, cryptocurrency |
| Person to person transactions | Can be transmitted, encrypted, etc. |
| Physical limitations (transmission, storage, etc.) | Third-party record of every transaction |
| No cryptographic security | |

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Barak Nehoran (Postdoc, CS Department, Columbia University)

Columbia University in the City of New York

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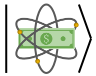
*Quantum Money*



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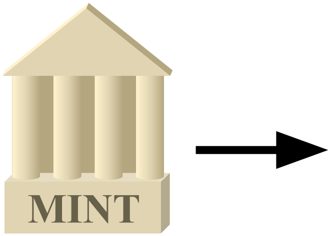

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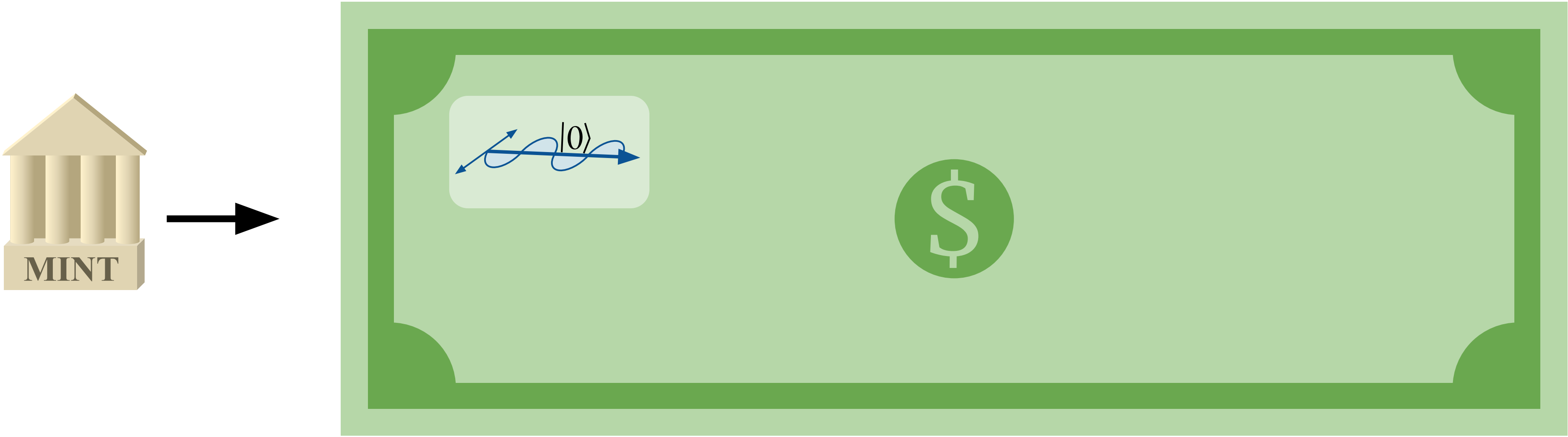

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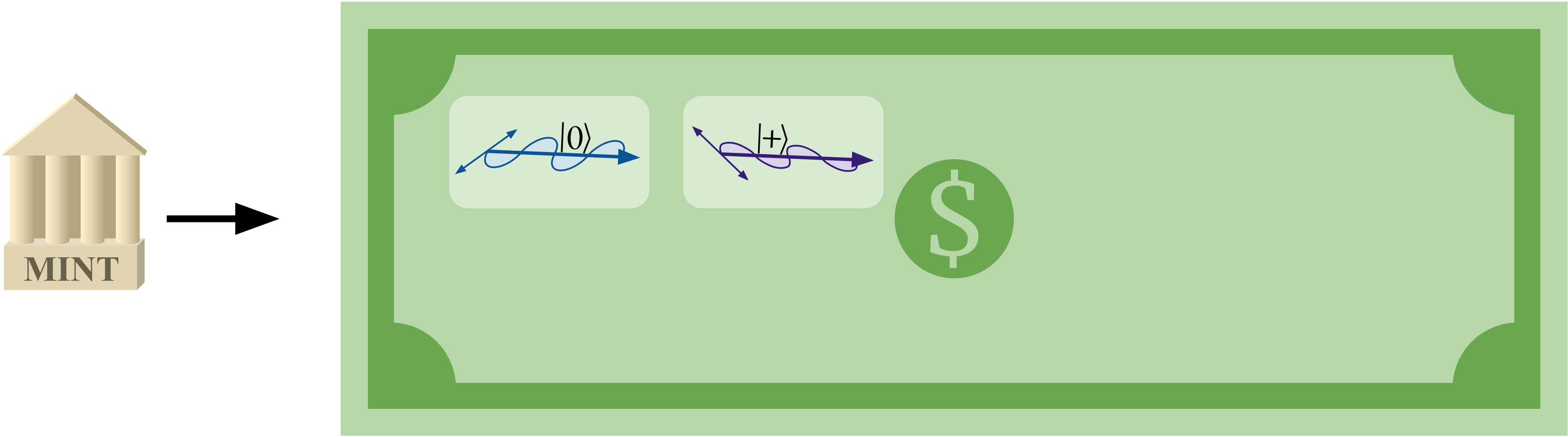

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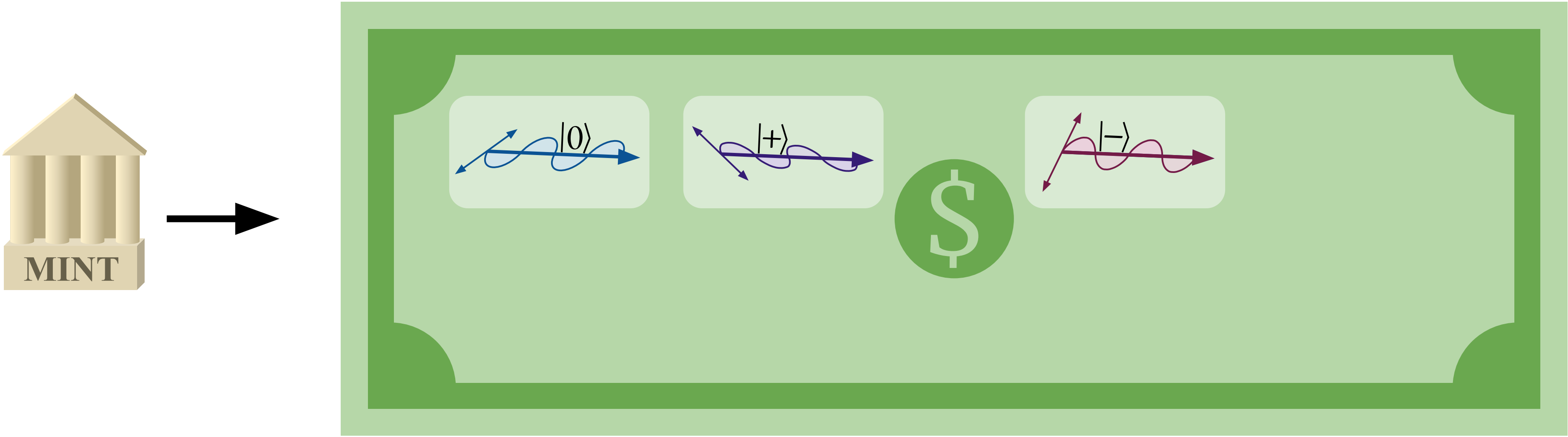

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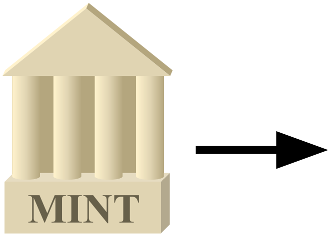


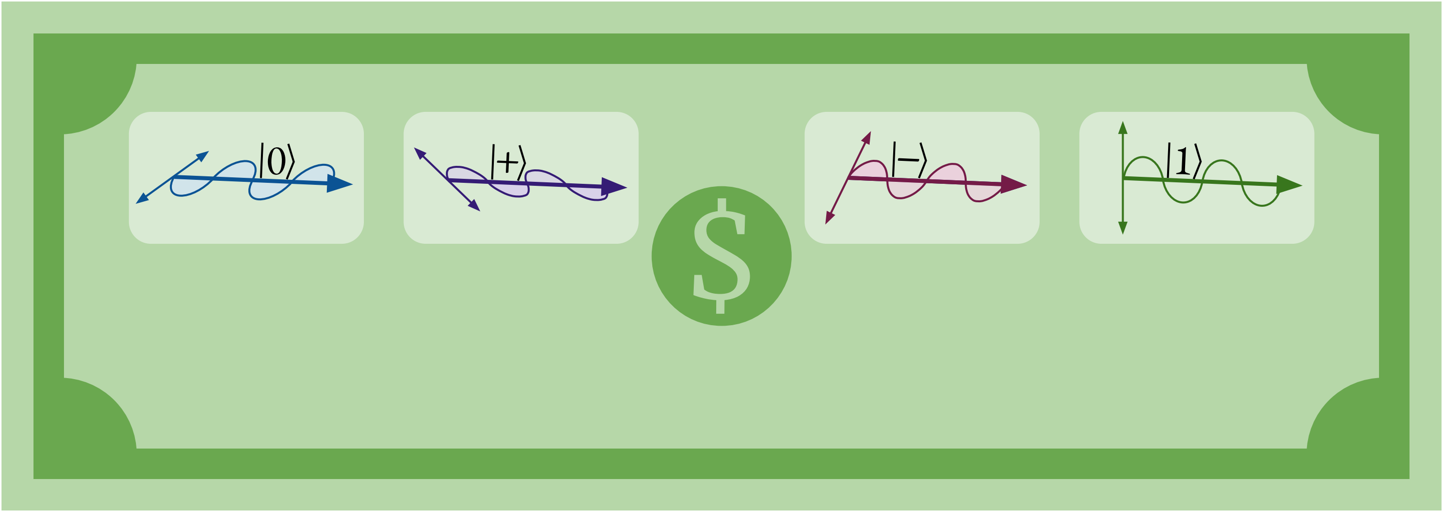

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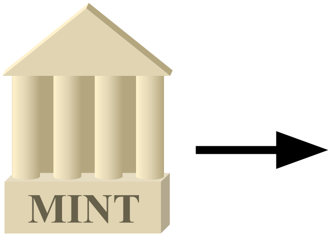


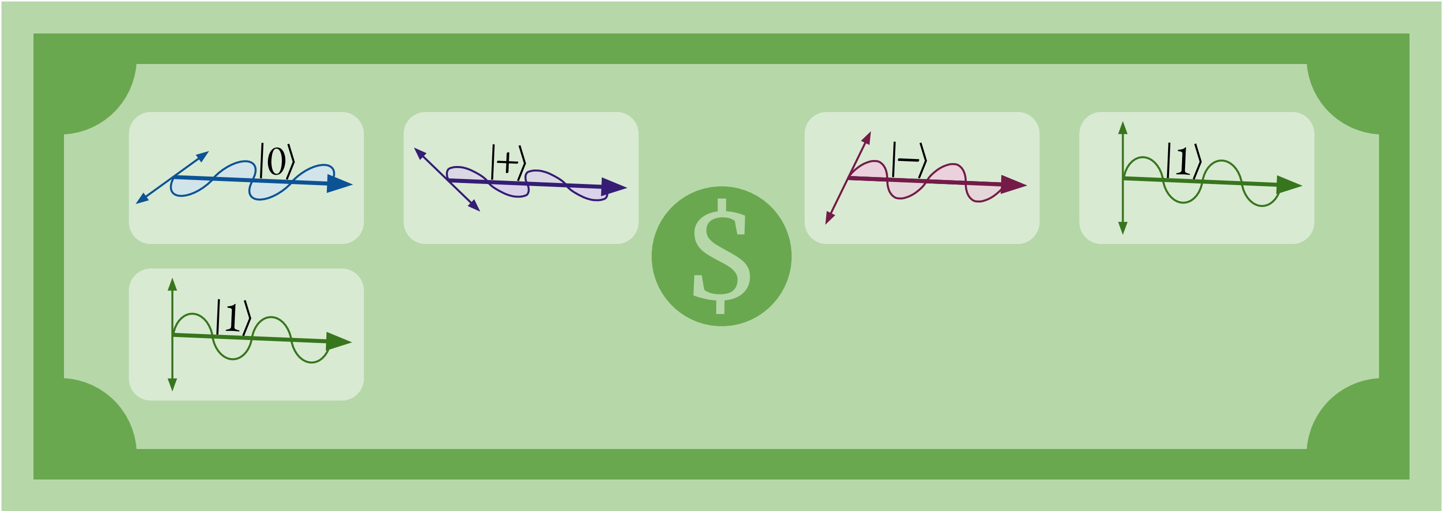

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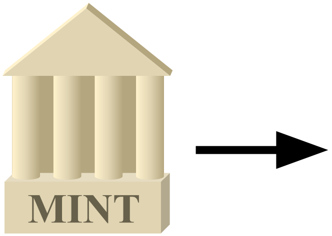


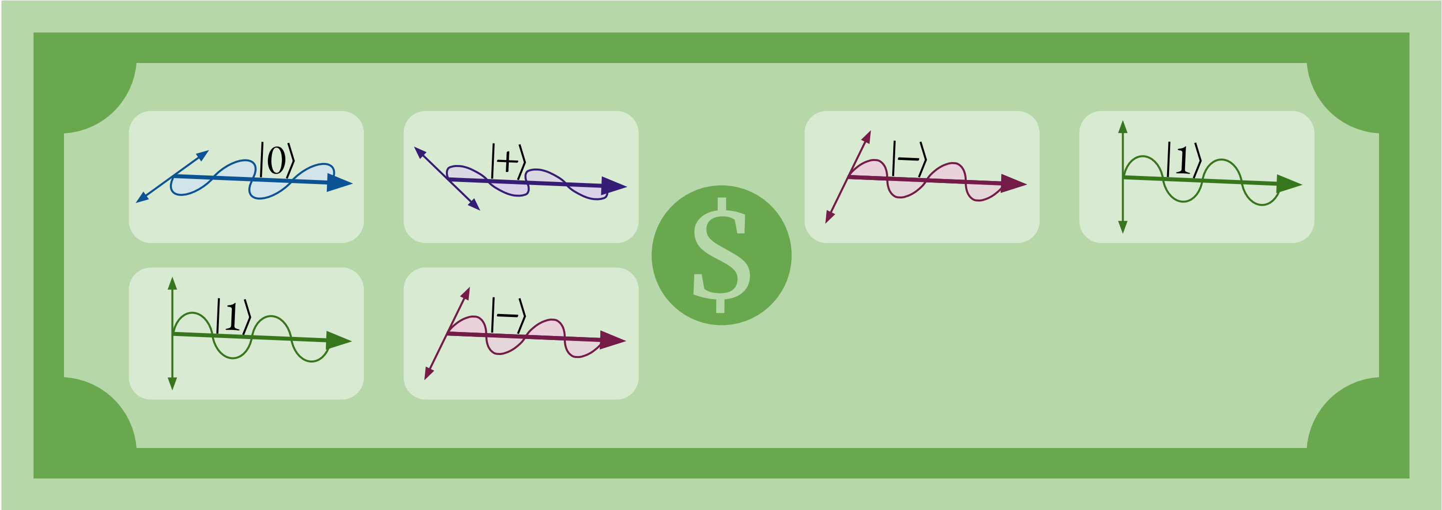

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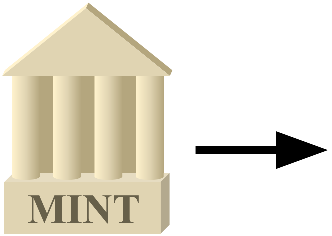


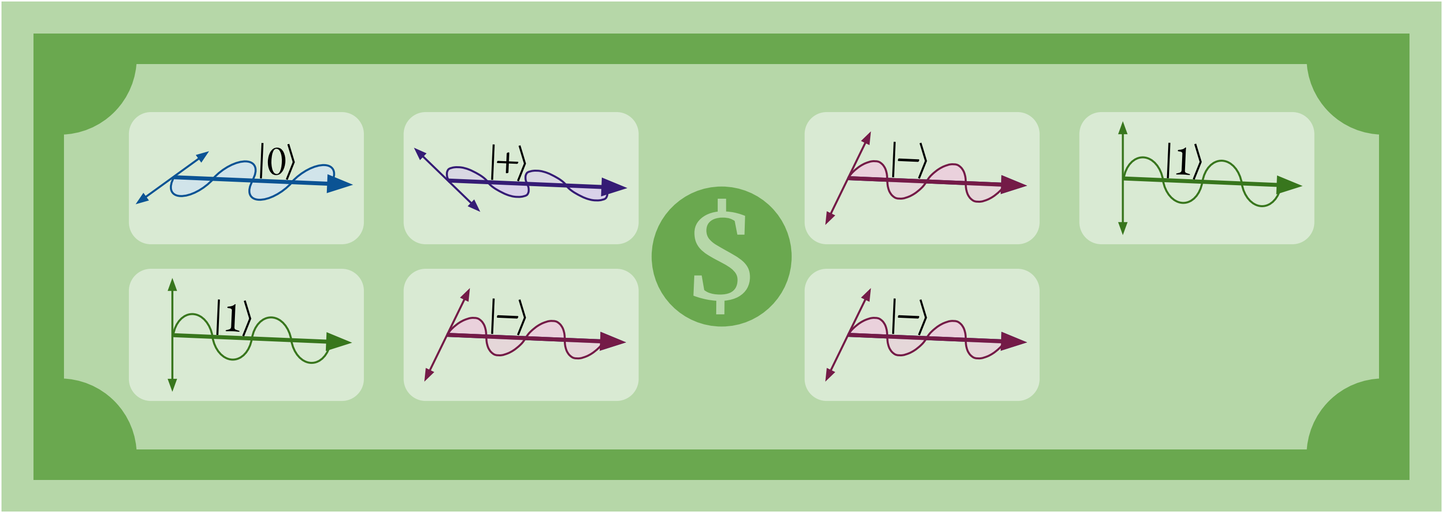

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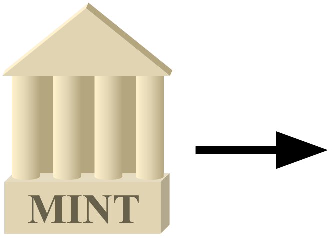


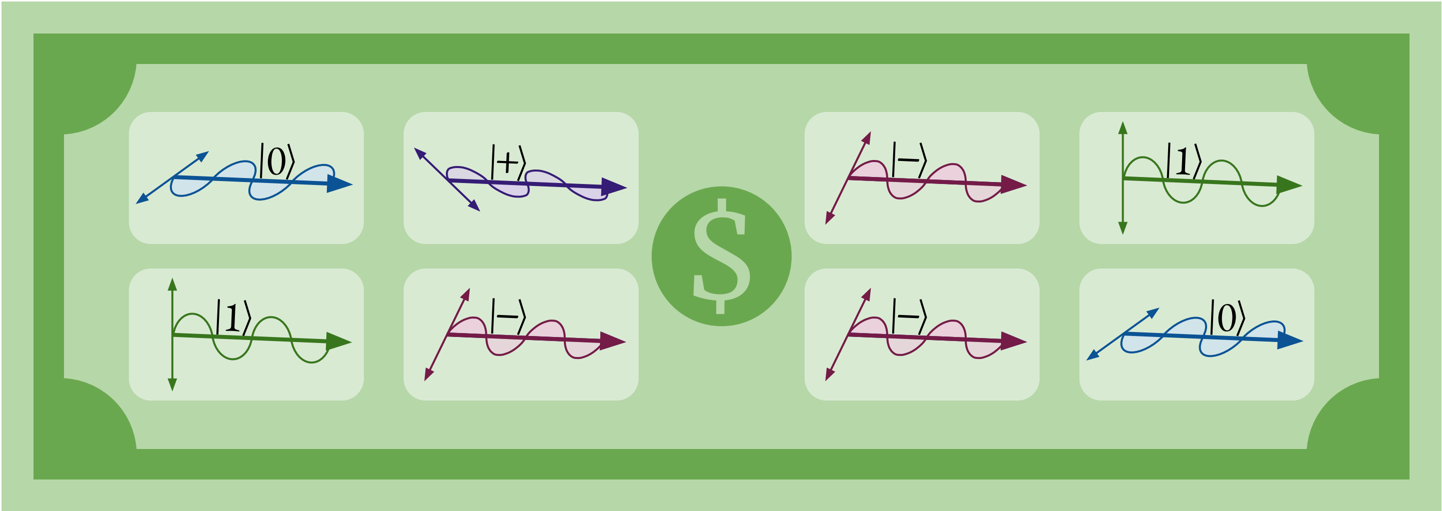

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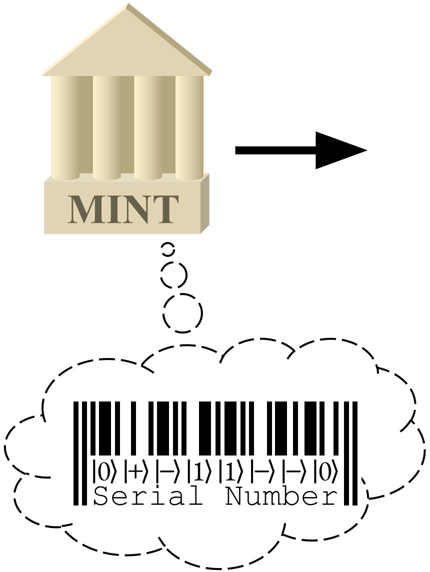


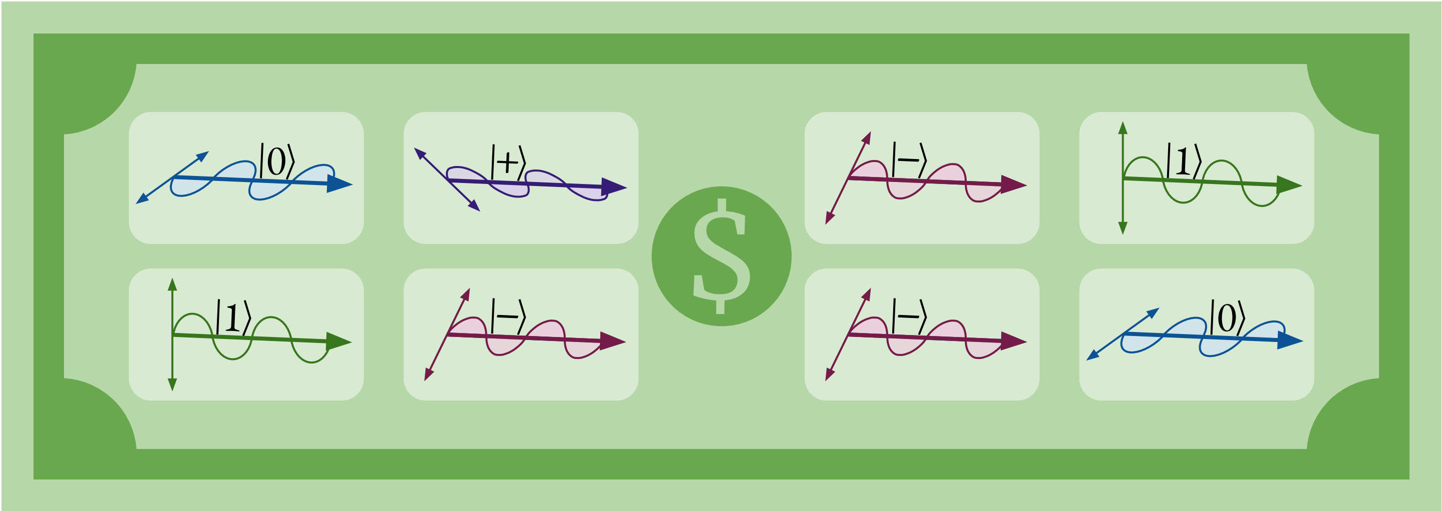

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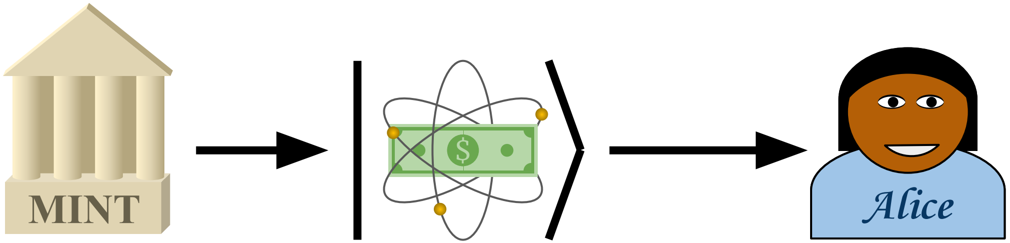

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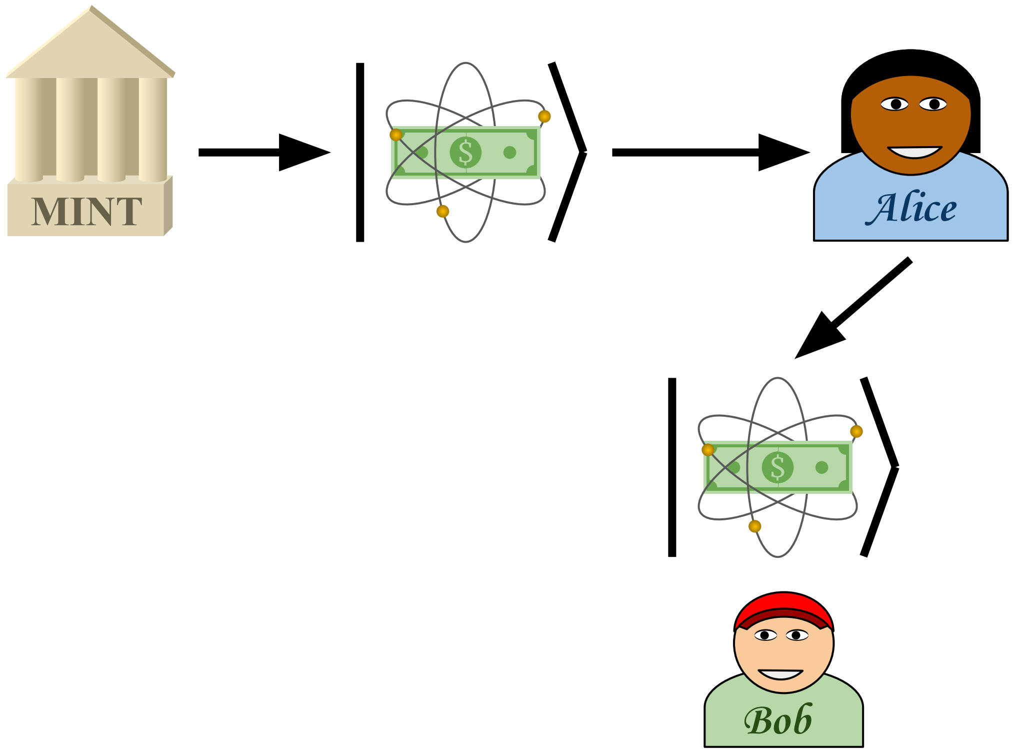

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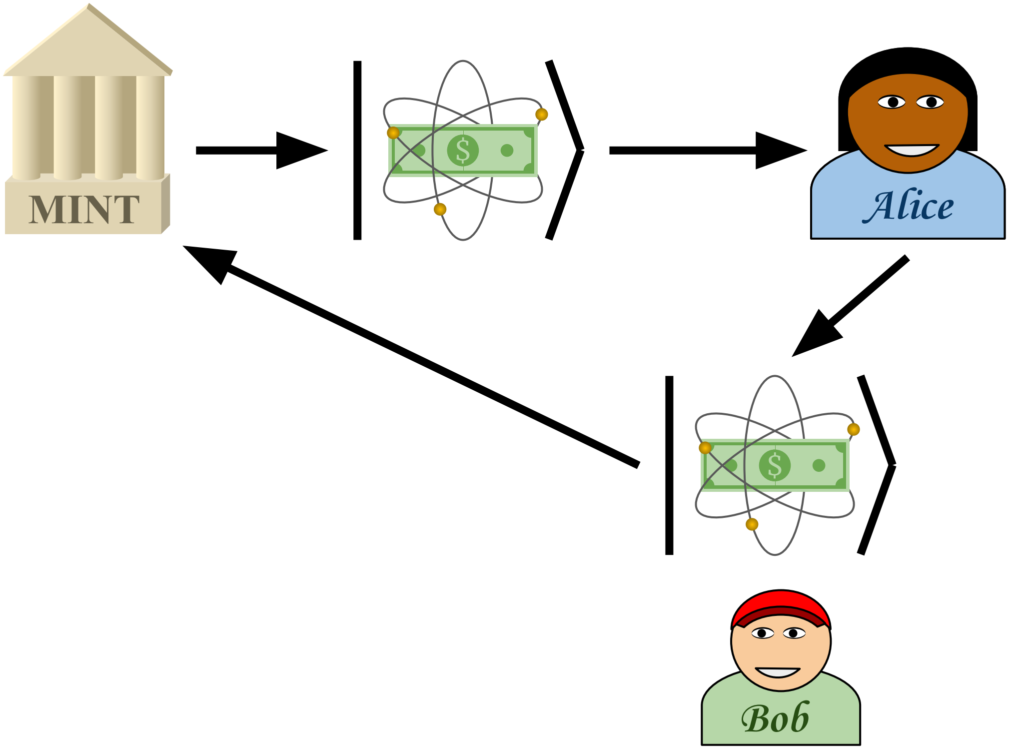

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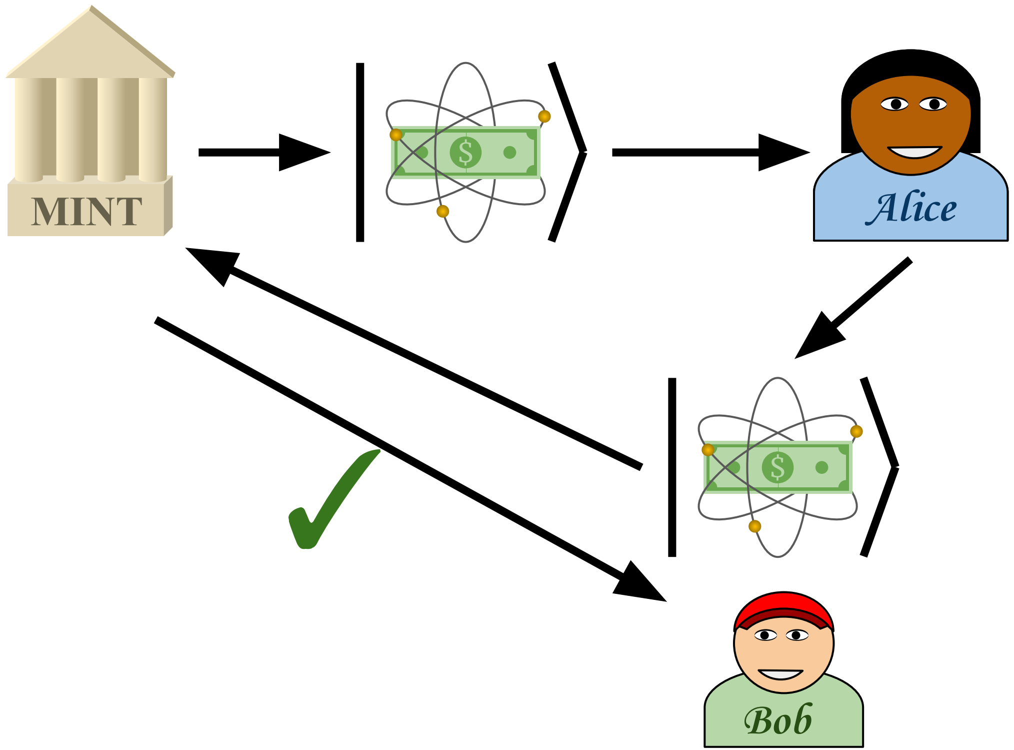

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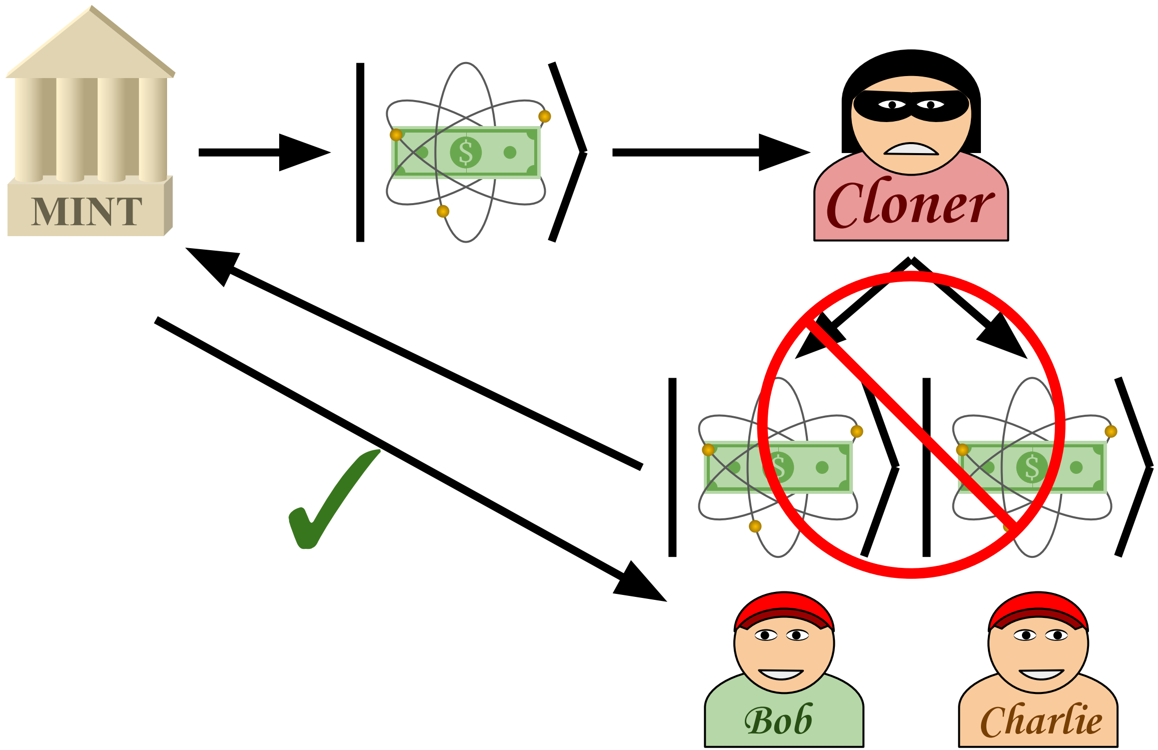

# What “cannot be cloned” means precisely

There is no two-qubit unitary $U$ such that for **every** single-qubit state $|\psi\rangle$,

$$U\Big(|\psi\rangle \otimes \underbrace{|0\rangle}_{\text{blank slate}}\Big) = |\psi\rangle \otimes |\psi\rangle\,.$$

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Note what is *not* claimed: a machine that copies only $|0\rangle$ and $|1\rangle$ is easy — that is just CNOT. The theorem rules out one machine that works for **all** states at once.

## Proof sketch

Suppose such a $U$ existed. Then

$$U|0\rangle|0\rangle = |0\rangle|0\rangle \qquad \text{and} \qquad U|+\rangle|0\rangle = |+\rangle|+\rangle .$$

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Unitaries preserve inner products. The inputs have inner product

$$\left(\langle 0| \otimes \langle 0|\right)\left(|+\rangle \otimes |0\rangle\right) = \frac{1}{\sqrt{2}},$$

while the outputs have

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$\frac{1}{\sqrt{2}} \neq \frac{1}{2}$, so no such $U$ exists. **Cloning would let you distinguish non-orthogonal states.**

Barak Nehoran (Postdoc, CS Department, Columbia University)

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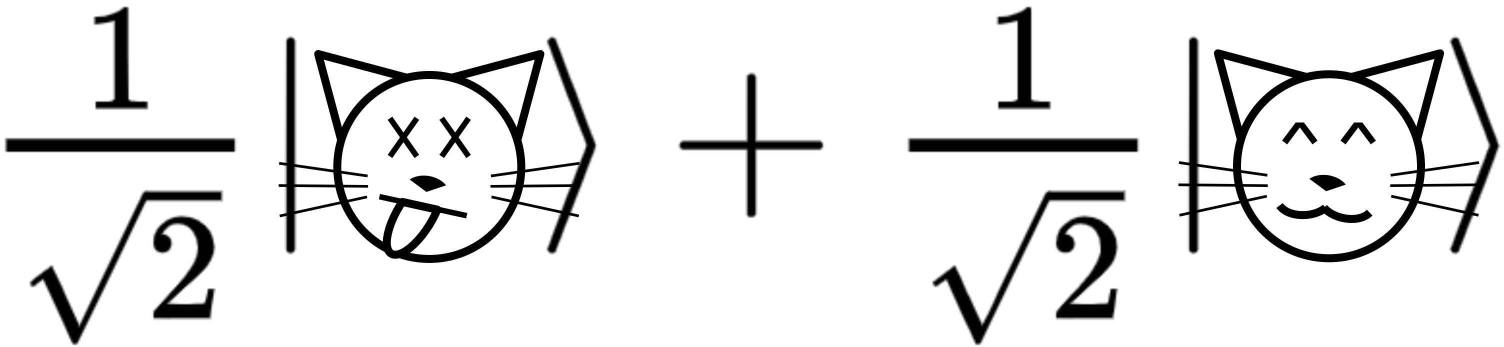

# 10 minute break