COMS 4281 - Introduction to Quantum Computing Fall 2026

Practice Worksheet 1 - Reversible Computing and Quantum Math

Problem 1: Reversible transformations and circuits

In class we covered how to take any boolean function $f : \{0,1\}^n \to \{0,1\}$ and convert it to its reversible equivalent R_f. Here we will convert a 2-bit function. Let $f : \{0,1\}^2 \to \{0,1\}$ be given by the truth table below.

x_1	x_2	$f(x_1,x_2)$
0	0	1
0	1	0
1	0	0
1	1	1

So $f(x_1,x_2) = 1$ when the two input bits are equal, otherwise known as $\neg$XOR. Define the reversible function (3-bit) by

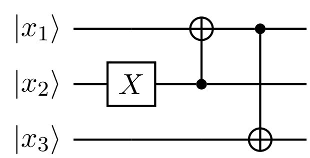
$$R_f : (x_1, x_2, b) \mapsto \big(x_1, x_2, b \oplus f(x_1,x_2)\big).$$

(a) Fill in the truth table of R_f for each 3-bit input (x_1, x_2, b), then write the corresponding output. Use the table below.

(b) Explain in a sentence why the mapping must be a permutation of $\{0,1\}^3$.

(c) Construct the corresponding 3-bit reversible circuit using CNOT and X gates that implements R_f. *Recall:* A CNOT with control s and target t replaces t by $t \oplus s$, leaving s unchanged.

(d) Below is a reversible circuit comprised of CNOT and X gates. It defines a reversible transformation on 3 bits. Write out the truth table of the transformation (i.e., what does every 3-bit input map to?).



Part (a)

Input (x_1, x_2, b)	Output of R_f
000	
001	
010	
011	
100	
101	
110	
111	

Part (d)

Input (x_1, x_2, x_3)	Circuit output
000	
001	
010	
011	
100	
101	
110	
111	

Optional extension. What is the inverse circuit for part (d)? Specify the order of the gates.

Problem 2: Matrix notation

Throughout this problem, use the ordered standard basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and the abbreviations

$$|a\rangle \otimes |b\rangle = |a\rangle |b\rangle = |a, b\rangle = |ab\rangle .$$

Let $F : \{0,1\}^2 \to \{0,1\}^2$ be the reversible function given by the truth table below.

x_1	x_2	x_1'	x_2'
0	0	1	1
0	1	0	1
1	0	1	0
1	1	0	0

Let R be its matrix representation, so that $R|x_1, x_2\rangle = |x_1', x_2'\rangle$.

(a) Write R as a 4×4 matrix.

(b) What is the 4-dimensional column vector corresponding to $R|1,1\rangle$? Write $|1,1\rangle = |1\rangle \otimes |1\rangle$ as a column vector first. What is the ket state corresponding to $R|1,1\rangle$?

(c) Construct the corresponding 2-bit reversible circuit using CNOT and X gates that implements F.

(d) Is R unitary? Explain your answer.

(e) What is R^{-1}, i.e., its inverse, as a matrix? How does the inverse behave on standard basis states?

Problem 3: From kets to vectors, and back

First, let's make sure we understand how to translate from ket notation to vector notation, and vice versa.

Consider the following two states, which together form the "Hadamard basis" for a qubit:

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) \qquad |-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big).$$

(a) Express $|0\rangle$ and $|1\rangle$ as linear combinations of $|+\rangle$ and $|-\rangle$. That is, write the standard basis vectors in the Hadamard basis.

(b) Consider the following vector:

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Write $|\psi\rangle$ as a linear combination of $|0\rangle, |1\rangle$. In other words, express $|\psi\rangle$ in terms of the standard basis.

(c) Write $|\psi\rangle$ as a linear combination of $|+\rangle, |-\rangle$. In other words, express $|\psi\rangle$ in terms of the Hadamard basis.

(d) Check that $|\psi\rangle$ is normalized. What are the measurement statistics of measuring $|\psi\rangle$ in the standard basis?

Problem 4: Applying the Hadamard gate

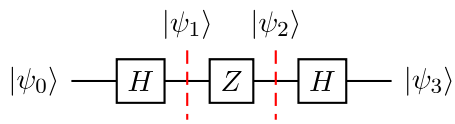
Here we're going to get some practice applying gates to a single qubit both using ket notation and the matrix/vector notation. We're going to see that these two notations are consistent with each other.

Consider the circuit below, with input state

$$|\psi_0\rangle = \frac{1}{\sqrt{5}}\big(|0\rangle + 2\,|1\rangle\big).$$

Recall the gates H and Z:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$



Read the circuit from left to right: apply H, then Z, then H. There are no measurements between the gates.

(a) Write out the states $|\psi_n\rangle$ shown in the circuit diagram in ket notation, for $n = 1, 2, 3$.

(b) Write the 2×2 matrix corresponding to this circuit. Is it unitary? What is its inverse, as a matrix?

(c) What are the measurement statistics of measuring $|\psi_1\rangle$? What are the measurement statistics of measuring $|\psi_3\rangle$? Compare with the measurement statistics of measuring $|\psi_0\rangle$ directly. Each comparison is a separate run, stopping at the state being measured.