COMS 4281 - Introduction to Quantum Computing Fall 2026

Practice Worksheet 1 - Reversible Computing and Quantum Math

Solutions

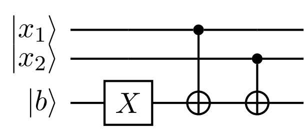
Problem 1: Reversible transformations and circuits

(a) $f = 1$ for $x_1x_2 \in \{00, 11\}$ and 0 otherwise. R_f flips b on those two inputs and does nothing otherwise. The truth tables for parts (a) and (d) are shown together below.

Input	Part (a): R_f output	Part (d): circuit output
000	001	111
001	000	110
010	010	000
011	011	001
100	100	010
101	101	011
110	111	101
111	110	100

(b) It is a permutation because R_f leaves x_1, x_2 unchanged and maps $b \mapsto b \oplus f(x_1, x_2)$, which is a bijection on the third bit for each fixed pair x_1, x_2.

(c) Since $f(x_1, x_2) = 1 \oplus x_1 \oplus x_2$, apply X to b, then $\text{CNOT}_{x_1 \to b}$ and $\text{CNOT}_{x_2 \to b}$. The target becomes $b \oplus 1 \oplus x_1 \oplus x_2 = b \oplus f(x_1, x_2)$; the input wires do not change.



(d) Reading the circuit from left to right, the wire values are

$$\begin{aligned}(x_1, x_2, x_3) &\longmapsto (x_1, x_2 \oplus 1, x_3)\\ &\longmapsto (x_1 \oplus x_2 \oplus 1, x_2 \oplus 1, x_3)\\ &\longmapsto (x_1 \oplus x_2 \oplus 1, x_2 \oplus 1, x_3 \oplus x_1 \oplus x_2 \oplus 1).\end{aligned}$$

The corresponding truth table is in the last column of the table above.

Optional extension. Each gate is self-inverse, so reverse the order of the gates: first apply $\text{CNOT}_{1\to3}$, then $\text{CNOT}_{2\to1}$, then X_2. The wire indices refer to fixed positions, and each gate acts on the current wire values.

Problem 2: Matrix notation

(a) The columns are the output vectors for inputs $00, 01, 10, 11$, in that order:

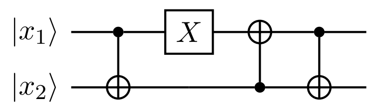
$$R = \begin{pmatrix} | & | & | & | \\ |11\rangle & |01\rangle & |10\rangle & |00\rangle \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(b)

$$|1,1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad R\,|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle\,.$$

(c) One circuit, in chronological order, is

$$\mathrm{CNOT}_{1\to 2}, \qquad X_1, \qquad \mathrm{CNOT}_{2\to 1}, \qquad \mathrm{CNOT}_{1\to 2}.$$



Tracing the wire values gives

$$\begin{aligned} (x_1, x_2) &\longmapsto (x_1, x_1 \oplus x_2) \\ &\longmapsto (x_1 \oplus 1, x_1 \oplus x_2) \\ &\longmapsto (x_2 \oplus 1, x_1 \oplus x_2) \\ &\longmapsto (x_2 \oplus 1, x_1 \oplus 1). \end{aligned}$$

This agrees with all four rows of the truth table. Other equivalent circuits are valid; a minimum gate count is not required.

(d) Yes. The columns are distinct standard basis vectors, hence orthonormal. Therefore $R^\dagger R = I$.

(e) R is real and symmetric, so

$$R^{-1} = R^\dagger = R^T = R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

On the standard basis, the inverse swaps $|00\rangle$ and $|11\rangle$, and fixes $|01\rangle$ and $|10\rangle$.

Problem 3: From kets to vectors, and back

(a) Adding and subtracting the definitions of $|+\rangle$ and $|-\rangle$ gives

$$|0\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle + |-\rangle\big), \qquad |1\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle - |-\rangle\big).$$

(b)

$$|\psi\rangle = \frac{1}{\sqrt{5}}\,|0\rangle + \frac{2i}{\sqrt{5}}\,|1\rangle\,.$$

(c) Substituting the expressions from part (a),

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{10}}\big(|+\rangle + |-\rangle\big) + \frac{2i}{\sqrt{10}}\big(|+\rangle - |-\rangle\big)\\ &= \frac{1+2i}{\sqrt{10}}\,|+\rangle + \frac{1-2i}{\sqrt{10}}\,|-\rangle\,.\end{aligned}$$

This is a new expression for the same state; rewriting it in a different basis is not a physical gate application.

(d) The normalization is

$$\left|\frac{1}{\sqrt{5}}\right|^2 + \left|\frac{2i}{\sqrt{5}}\right|^2 = \frac{1}{5} + \frac{4}{5} = 1.$$

In the standard basis, the measurement probabilities are

$$\Pr(0) = \frac{1}{5}, \qquad \Pr(1) = \frac{4}{5}.$$

The magnitude squared of $2i/\sqrt{5}$ is $4/5$, not $-4/5$. The coefficients in part (c) are not the standard-basis amplitudes.

Problem 4: Applying the Hadamard gate

(a) Starting from $|\psi_0\rangle = (|0\rangle + 2\,|1\rangle)/\sqrt{5}$, linearity gives

$$\begin{aligned}
|\psi_1\rangle = H\,|\psi_0\rangle &= \frac{1}{\sqrt{10}}\big[\big(|0\rangle + |1\rangle\big) + 2\big(|0\rangle - |1\rangle\big)\big] \\
&= \frac{1}{\sqrt{10}}\big(3\,|0\rangle - |1\rangle\big), \\
|\psi_2\rangle = Z\,|\psi_1\rangle &= \frac{1}{\sqrt{10}}\big(3\,|0\rangle + |1\rangle\big), \\
|\psi_3\rangle = H\,|\psi_2\rangle &= \frac{1}{\sqrt{20}}\big[3\big(|0\rangle + |1\rangle\big) + \big(|0\rangle - |1\rangle\big)\big] \\
&= \frac{1}{\sqrt{20}}\big(4\,|0\rangle + 2\,|1\rangle\big) = \frac{1}{\sqrt{5}}\big(2\,|0\rangle + |1\rangle\big).
\end{aligned}$$

(b) The first gate acts on the right in the product. Thus the circuit matrix is

$$\begin{aligned}
U = HZH &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = X.
\end{aligned}$$

It is unitary: $U^\dagger U = X^2 = I$. Its inverse is

$$U^{-1} = U^\dagger = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = X.$$

This also checks the result of part (a): U swaps the two amplitudes of $|\psi_0\rangle$.

(c) Squaring the magnitudes of the standard-basis amplitudes gives

State measured	Pr(0)	Pr(1)
$\|\psi_0\rangle$	1/5	4/5
$\|\psi_1\rangle$	9/10	1/10
$\|\psi_3\rangle$	4/5	1/5

Hadamard does not always produce equal measurement probabilities. For this input, the first H gives 9/10 and 1/10. The entire circuit swaps the input amplitudes, so it swaps their measurement probabilities.