

COMS 4281 — Introduction to Quantum Computing

Practice Worksheet 2: Qubit Math

Show your calculations and briefly justify your conclusions.

Notation. We use the computational basis $|0\rangle, |1\rangle$ and

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Two-qubit column vectors are ordered as $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Problem 1: Complex amplitudes and inner products

Consider the vector

$$|v\rangle = (1 + i)|0\rangle + 2|1\rangle.$$

- (a) Find the positive real number c such that $|u\rangle = c|v\rangle$ is normalized. Write $|u\rangle$ as a column vector.
- (b) Write $\langle u|$ in both bra notation and row-vector notation, and verify $\langle u | u \rangle = 1$. What value would you get instead if you transposed the column vector without conjugating its entries?
- (c) Compute $\langle + | u \rangle$ and $\langle u | + \rangle$. How are the two answers related?
- (d) Let $|w\rangle = \frac{2|0\rangle - (1 - i)|1\rangle}{\sqrt{6}}$. Check that $|w\rangle$ is normalized and determine whether it is orthogonal to $|u\rangle$.

Problem 2: Same probabilities, different states?

Consider the two normalized states

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle, \quad |\phi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle.$$

Each experiment below starts with a fresh copy of the indicated input state.

- (a) Suppose you measure immediately in the computational basis. Calculate the probabilities of outcomes 0 and 1 for each input. Can these measurement statistics tell the two inputs apart?

- (b) Now apply H before measuring. Compute $H|\psi\rangle$ and $H|\phi\rangle$, and then the probabilities of outcomes 0 and 1 for each input.

- (c) Measuring after H is equivalent to measuring the original state in the $\{|+\rangle, |-\rangle\}$ basis. Verify this for input $|\psi\rangle$ by computing $|\langle+|\psi\rangle|^2$ and $|\langle-|\psi\rangle|^2$ directly. Does a single outcome of this measurement always identify whether the input was $|\psi\rangle$ or $|\phi\rangle$? Explain using part (b).

- (d) Compare $|\psi\rangle$, $|\phi\rangle$, and $i|\psi\rangle$. Which pairs differ only by a global phase? Which differ by a relative phase? For each experiment above, would $i|\psi\rangle$ have the same outcome probabilities as $|\psi\rangle$? Explain without repeating all the arithmetic.

Problem 3: Tensor products and entanglement

Let

$$|a\rangle = \frac{|0\rangle + 2|1\rangle}{\sqrt{5}}, \quad |b\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad |P\rangle = |a\rangle \otimes |b\rangle.$$

- (a) Expand $|P\rangle$ in the computational basis and write its four-component column vector. Then calculate the probabilities of the four outcomes when both qubits are measured in that basis.

- (b) Consider the state

$$|E\rangle = \frac{|00\rangle + i|01\rangle + 2|10\rangle - 2i|11\rangle}{\sqrt{10}}.$$

Check that it is normalized. Compare its four computational-basis measurement probabilities with those of $|P\rangle$.

- (c) A pure two-qubit state is a *product state* if it can be written as $|r\rangle \otimes |s\rangle$ for single-qubit states $|r\rangle$ and $|s\rangle$; otherwise it is *entangled*. Determine whether $|E\rangle$ is a product state.

Hint: Expand

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

and match all four coefficients to those of $|E\rangle$. Can the four resulting equations hold simultaneously?

- (d) A student claims: “If two pure states give the same computational-basis measurement probabilities, then either both are product states or both are entangled.” Is the claim correct? Use your calculations to justify your answer.

Problem 4: Measuring only one qubit

Alice holds the first qubit and Bob holds the second qubit of

$$|\Omega\rangle = \frac{|00\rangle + |01\rangle + 2|10\rangle - |11\rangle}{\sqrt{7}}.$$

Alice measures her qubit in the computational basis. Write $A \in \{0, 1\}$ for her outcome. Bob subsequently measures his qubit in the $\{|+\rangle, |-\rangle\}$ basis; write $B \in \{+, -\}$ for his outcome.

- (a) Complete the table. Give *normalized* post-measurement states, conditioned on Alice's outcome and before Bob measures.

Alice's outcome	Probability	Full two-qubit state	Bob's one-qubit state
$A = 0$			
$A = 1$			

- (b) Calculate Bob's conditional outcome probabilities and fill in the table.

Alice's outcome	$\Pr(B = + \mid A = a)$	$\Pr(B = - \mid A = a)$
$a = 0$		
$a = 1$		

- (c) Alice does not tell Bob her outcome. Compute Bob's overall probabilities $\Pr(B = +)$ and $\Pr(B = -)$ by combining the two cases. Explain why the two rows in part (b) should not be weighted equally.
- (d) **Optional extension.** Suppose Alice does *not* measure. Compute $(I \otimes H)|\Omega\rangle$, where I is the identity on Alice's qubit. Use it to find Bob's $+$ and $-$ probabilities, and compare with part (c). Can Bob's statistics for this measurement reveal whether Alice measured?