COMS 4281 - Introduction to Quantum Computing Fall 2026

Practice Worksheet 2 - Qubit Math

Solutions

Problem 1: Complex amplitudes and inner products

(a) Since $|1+i|^2 + |2|^2 = 2 + 4 = 6$, normalization requires $6c^2 = 1$. Thus

$$c = \frac{1}{\sqrt{6}}, \qquad |u\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

(b) Conjugate the coefficients when turning a ket into a bra:

$$\langle u| = \frac{1-i}{\sqrt{6}} \langle 0| + \frac{2}{\sqrt{6}} \langle 1| = \frac{1}{\sqrt{6}} \begin{pmatrix} 1-i & 2 \end{pmatrix}.$$

Then

$$\langle u|u\rangle = \frac{(1-i)(1+i)+4}{6} = \frac{2+4}{6} = 1.$$

Transposing without conjugating would instead give

$$\frac{(1+i)^2+4}{6} = \frac{4+2i}{6} = \frac{2}{3} + \frac{i}{3},$$

which is not the inner product.

(c)

$$\langle +|u\rangle = \frac{1+i+2}{\sqrt{12}} = \frac{3+i}{\sqrt{12}}, \qquad \langle u|+\rangle = \frac{3-i}{\sqrt{12}}.$$

The two answers are complex conjugates.

(d) For $|w\rangle = (2|0\rangle - (1-i)|1\rangle)/\sqrt{6}$,

$$\langle w|w\rangle = \frac{4 + |1-i|^2}{6} = \frac{4+2}{6} = 1.$$

Also,

$$\langle u|w\rangle = \frac{2(1-i) - 2(1-i)}{6} = 0.$$

So $|w\rangle$ is normalized and orthogonal to $|u\rangle$.

Problem 2: Same probabilities, different states?

The two inputs are $|\psi\rangle = (3\,|0\rangle + 4\,|1\rangle)/5$ and $|\phi\rangle = (3\,|0\rangle - 4\,|1\rangle)/5$.

(a) Squaring the magnitudes of the amplitudes gives the same probabilities for both states:

Input	Pr(0)	Pr(1)
$\|\psi\rangle$	9/25	16/25
$\|\phi\rangle$	9/25	16/25

These measurement statistics cannot tell the inputs apart.

(b) Apply $H\,|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $H\,|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$:

$$H\,|\psi\rangle = \frac{3(|0\rangle + |1\rangle) + 4(|0\rangle - |1\rangle)}{5\sqrt{2}} = \frac{7\,|0\rangle - |1\rangle}{\sqrt{50}},$$
$$H\,|\phi\rangle = \frac{3(|0\rangle + |1\rangle) - 4(|0\rangle - |1\rangle)}{5\sqrt{2}} = \frac{-\,|0\rangle + 7\,|1\rangle}{\sqrt{50}}.$$

Therefore,

Input before H	Pr(0)	Pr(1)
$\|\psi\rangle$	49/50	1/50
$\|\phi\rangle$	1/50	49/50

(c) Directly,

$$\langle +|\psi\rangle = \frac{3+4}{5\sqrt{2}} = \frac{7}{\sqrt{50}}, \qquad \langle -|\psi\rangle = \frac{3-4}{5\sqrt{2}} = -\frac{1}{\sqrt{50}}.$$

Their squared magnitudes are 49/50 and 1/50, agreeing with part (b). A single outcome does not always identify the input: either outcome has nonzero probability for either input.

(d) $|\psi\rangle$ and $i\,|\psi\rangle$ differ only by a global phase. $|\psi\rangle$ and $|\phi\rangle$ have a relative minus sign between their two amplitudes. After removing the global factor i, the same relative-phase difference remains between $i\,|\psi\rangle$ and $|\phi\rangle$.

For any of the measurements above, multiplying the input by i multiplies every outcome amplitude by i. Since $|iz|^2 = |z|^2$, $i\,|\psi\rangle$ gives exactly the same probabilities as $|\psi\rangle$.

Problem 3: Tensor products and entanglement

(a) Expanding the tensor product gives

$$|P\rangle = \frac{|0\rangle + 2\,|1\rangle}{\sqrt{5}} \otimes \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$$
$$= \frac{|00\rangle + i\,|01\rangle + 2\,|10\rangle + 2i\,|11\rangle}{\sqrt{10}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ i \\ 2 \\ 2i \end{pmatrix}.$$

The measurement probabilities are

Outcome	00	01	10	11
Probability	1/10	1/10	2/5	2/5

(b) For

$$|E\rangle = \frac{|00\rangle + i\,|01\rangle + 2\,|10\rangle - 2i\,|11\rangle}{\sqrt{10}},$$

the squared magnitudes sum to $(1 + 1 + 4 + 4)/10 = 1$. The only change from $|P\rangle$ is a minus sign on the $|11\rangle$ amplitude. Thus all four computational-basis probabilities are unchanged.

(c) Suppose

$$|E\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\gamma\,|0\rangle + \delta\,|1\rangle).$$

Matching coefficients gives

$$\alpha\gamma = \frac{1}{\sqrt{10}}, \quad \alpha\delta = \frac{i}{\sqrt{10}}, \quad \beta\gamma = \frac{2}{\sqrt{10}}, \quad \beta\delta = -\frac{2i}{\sqrt{10}}.$$

The first equation implies $\alpha \neq 0$, and the first two imply $\delta = i\gamma$. But then

$$\beta\delta = i\beta\gamma = \frac{2i}{\sqrt{10}},$$

contradicting the fourth equation. Therefore $|E\rangle$ is entangled.

(d) The claim is false. $|P\rangle$ is a product state by construction, whereas $|E\rangle$ is entangled, but their computational-basis measurement probabilities are identical.

Problem 4: Measuring only one qubit

Start with

$$|\Omega\rangle = \frac{|00\rangle + |01\rangle + 2\,|10\rangle - |11\rangle}{\sqrt{7}}.$$

(a) For each Alice outcome, delete the terms with the other first bit. Add the squared magnitudes of the remaining amplitudes to find its probability, then divide the remaining vector by the square root of that probability.

For $A = 0$, the probability is $(1 + 1)/7 = 2/7$. For $A = 1$, it is $(4 + 1)/7 = 5/7$. Thus

Alice's outcome	Probability	Full two-qubit state	Bob's state
$A = 0$	$2/7$	$\dfrac{\|00\rangle + \|01\rangle}{\sqrt{2}}$	$\|+\rangle$
$A = 1$	$5/7$	$\dfrac{2\,\|10\rangle - \|11\rangle}{\sqrt{5}}$	$\dfrac{2\,\|0\rangle - \|1\rangle}{\sqrt{5}}$

(b) If $A = 0$, Bob has $|+\rangle$ and gets $+$ with certainty. If $A = 1$, his amplitudes in the Hadamard basis are

$$\frac{2-1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \quad \text{for } +, \qquad \frac{2+1}{\sqrt{10}} = \frac{3}{\sqrt{10}} \quad \text{for } -.$$

Squaring gives

Alice's outcome	$\Pr(B = + \mid A = a)$	$\Pr(B = - \mid A = a)$
$a = 0$	1	0
$a = 1$	$1/10$	$9/10$

(c) Weight the two cases by Alice's outcome probabilities:

$$\Pr(B = +) = \frac{2}{7} \cdot 1 + \frac{5}{7} \cdot \frac{1}{10} = \frac{5}{14}, \qquad \Pr(B = -) = \frac{2}{7} \cdot 0 + \frac{5}{7} \cdot \frac{9}{10} = \frac{9}{14}.$$

The rows should not be weighted equally because Alice's outcomes occur with probabilities $2/7$ and $5/7$, not $1/2$ each.

(d) **Optional extension.** Applying H to Bob's qubit gives

$$\begin{aligned}(I \otimes H)\,|\Omega\rangle &= \frac{|0\rangle\,(2\,|0\rangle) + |1\rangle\,(|0\rangle + 3\,|1\rangle)}{\sqrt{14}} \\ &= \frac{2\,|00\rangle + |10\rangle + 3\,|11\rangle}{\sqrt{14}}.\end{aligned}$$

The probability that the second bit is 0 is $(4 + 1)/14 = 5/14$; the probability it is 1 is $9/14$. These correspond to Bob's $+$ and $-$ outcomes. They agree with part (c), so these statistics cannot reveal whether Alice measured.