

Practice Worksheet 3 - Composite Systems

Lecture 4: Entanglement and partial measurements

Show your reasoning. All measurements are in the computational basis. We write $|ab\rangle = |a\rangle_A \otimes |b\rangle_B$ and use the basis order $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Problem 1: Building a two-qubit state

Qubits A and B are prepared independently in the states

$$|+\rangle_A = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}, \quad |v\rangle_B = \frac{|0\rangle_B + 2|1\rangle_B}{\sqrt{5}}.$$

Let $|\psi\rangle = |+\rangle_A \otimes |v\rangle_B$.

(a) Expand $|\psi\rangle$ in the computational basis, then write its four-dimensional column vector.

(b) If both qubits are measured, what is the probability of each outcome?

Outcome (A, B)	00	01	10	11
Probability				

Problem 2: Proving entanglement or unentanglement

For each state, decide whether it is entangled or unentangled, and prove your answer. For an unentangled state, give a tensor product of normalized single-qubit states. For an entangled state, assume a factorization $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ and derive a contradiction by matching coefficients.

(a) $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$.

(b) $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$.

Problem 3: Which qubits are entangled?

Label the qubits 1, 2, 3 from left to right. A three-qubit state is *completely unentangled* if it is a tensor product of three single-qubit states. It may instead have an entangled pair and a third qubit that factors out.

In (a)–(c), identify the groups by displaying a factorization. If a pair remains, prove that it is entangled. Keep qubit labels explicit when regrouping tensor factors.

(a) $\frac{1}{\sqrt{8}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle).$

(b) $\frac{1}{\sqrt{2}}(|000\rangle + |110\rangle).$

(c) $\frac{1}{2}(|000\rangle + |010\rangle - |101\rangle - |111\rangle).$

(d) **Optional challenge: the GHZ state.** Consider

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Prove that no single qubit factors out from the other two: no factorization exists across $1 \mid 23$, $2 \mid 13$, or $3 \mid 12$.

Hint: For $1 \mid 23$, allow an arbitrary two-qubit state on 2, 3; do not assume those two qubits factor from each other.

Problem 4: Measuring only part of the system

Start with a fresh copy of

$$|\phi\rangle = \frac{|00\rangle + 2|01\rangle + |10\rangle - 2|11\rangle}{\sqrt{10}},$$

and measure only qubit B .

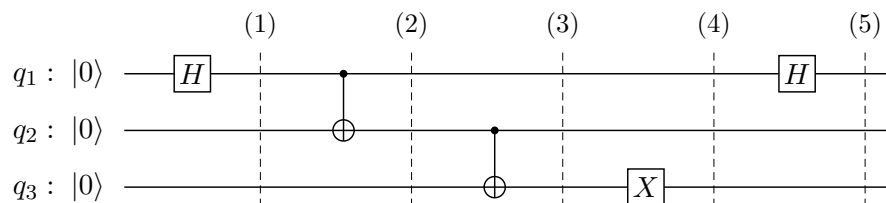
- (a) Fill in the table. For each outcome, find its probability, keep the consistent terms, and renormalize. Give both the joint state and the state of the unmeasured qubit A .

Outcome on B	Probability	Normalized joint state	Normalized state of A
0			
1			

- (b) Suppose the outcome on B is 1. If you now measure A , what are $\Pr[A = 0 \mid B = 1]$ and $\Pr[A = 1 \mid B = 1]$? What is the probability that the *full experiment* gives $B = 1$ followed by $A = 0$?
- (c) Instead, after obtaining $B = 1$, apply a Hadamard gate H to A before measuring it. What outcome occurs with certainty? How would the answer change if the initial state were the product state $|\psi\rangle$ from Problem 1, again conditioned on $B = 1$?
You may use $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$, $H|+\rangle = |0\rangle$, and $H|-\rangle = |1\rangle$.

Problem 5: Creating entanglement with a circuit

Start in $|000\rangle$ and apply the gates from left to right. Qubits 1, 2, 3 run from top to bottom; write kets in the order $|q_1q_2q_3\rangle$. The dashed lines mark checkpoints after each gate.



- (a) Write the normalized three-qubit state at each checkpoint as a superposition of computational-basis states. Keep track of relative signs.

$$|\psi_1\rangle =$$

$$|\psi_2\rangle =$$

$$|\psi_3\rangle =$$

$$|\psi_4\rangle =$$

$$|\psi_5\rangle =$$

- (b) Measure all three qubits in the final state $|\psi_5\rangle$. List all outcomes with nonzero probability, and give their probabilities. What is the probability that qubits 2 and 3 give the same bit?
- (c) On a fresh run of the circuit, measure *only qubit 1* in $|\psi_5\rangle$. For each possible outcome, give its probability and the normalized state of qubits 2, 3. Are those two qubits entangled after the measurement? Briefly justify your answer.