COMS 4281 - Introduction to Quantum Computing Fall 2026

Practice Worksheet 3 - Composite Systems

Solutions

Problem 1: Building a two-qubit state

(a) The first tensor factor is qubit A and the second is qubit B:

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + 2\,|1\rangle}{\sqrt{5}}$$
$$= \frac{|00\rangle + 2\,|01\rangle + |10\rangle + 2\,|11\rangle}{\sqrt{10}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

(b) Square the magnitudes of the four amplitudes:

Outcome (A, B)	00	01	10	11
Probability	1/10	2/5	1/10	2/5

Problem 2: Proving entanglement or unentanglement

(a) The state $(|00\rangle - |01\rangle + |10\rangle + |11\rangle)/2$ is entangled.

Suppose it were $(a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle)$. Matching coefficients would give

$$ac = \frac{1}{2}, \quad ad = -\frac{1}{2}, \quad bc = \frac{1}{2}, \quad bd = \frac{1}{2}.$$

Since $c \neq 0$, the equations $ac = bc$ imply $a = b$. But then $ad = bd$, contradicting $ad = -1/2$ and $bd = 1/2$.

(b) The state is unentangled:

$$\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |+\rangle \otimes |-\rangle.$$

Both single-qubit factors are normalized.

Problem 3: Which qubits are entangled?

(a) The eight-term state factors as

$$\underbrace{\frac{|0\rangle + |1\rangle}{\sqrt{2}}}_{\text{qubit 1}} \otimes \underbrace{\frac{|0\rangle + |1\rangle}{\sqrt{2}}}_{\text{qubit 2}} \otimes \underbrace{\frac{|0\rangle - |1\rangle}{\sqrt{2}}}_{\text{qubit 3}}.$$

Thus the state is completely unentangled.

(b)

$$\frac{|000\rangle + |110\rangle}{\sqrt{2}} = \underbrace{\frac{|00\rangle + |11\rangle}{\sqrt{2}}}_{\text{qubits 1,2}} \otimes \underbrace{|0\rangle}_{\text{qubit 3}}.$$

Qubit 3 factors out. Qubits 1 and 2 are entangled: a product factorization would require

$$ac = bd = \frac{1}{\sqrt{2}}, \qquad ad = bc = 0.$$

The first two nonzero coefficients force a, b, c, d all to be nonzero, contradicting $ad = 0$.

(c) In the original qubit order $1, 2, 3$, the state is

$$\frac{|000\rangle + |010\rangle - |101\rangle - |111\rangle}{2} = \frac{|0\rangle\,|+\rangle\,|0\rangle - |1\rangle\,|+\rangle\,|1\rangle}{\sqrt{2}}.$$

Qubit 2 factors out in state $|+\rangle$. Grouping qubits 1 and 3 together gives

$$\underbrace{\frac{|00\rangle - |11\rangle}{\sqrt{2}}}_{\text{qubits 1,3}} \otimes \underbrace{|+\rangle}_{\text{qubit 2}}.$$

The pair is entangled: matching coefficients would require $ac = 1/\sqrt{2}$, $bd = -1/\sqrt{2}$, and $ad = bc = 0$, the same contradiction as in part (b).

(d) **Optional challenge: the GHZ state.** Suppose a factorization across $1 \mid 23$ existed:

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}} = (a\,|0\rangle + b\,|1\rangle) \otimes (c_{00}\,|00\rangle + c_{01}\,|01\rangle + c_{10}\,|10\rangle + c_{11}\,|11\rangle).$$

Matching the $|000\rangle$ and $|111\rangle$ coefficients gives

$$ac_{00} = \frac{1}{\sqrt{2}}, \qquad bc_{11} = \frac{1}{\sqrt{2}}.$$

Thus $a \neq 0$ and $c_{11} \neq 0$. But the $|011\rangle$ coefficient must be $ac_{11} = 0$, a contradiction. The GHZ state is unchanged by permuting its qubits, so the same proof applies across $2 \mid 13$ and $3 \mid 12$.

Problem 4: Measuring only part of the system

The initial state is

$$|\phi\rangle = \frac{|00\rangle + 2\,|01\rangle + |10\rangle - 2\,|11\rangle}{\sqrt{10}}.$$

(a) For $B = 0$, keep the terms whose second bit is 0:

$$\frac{|00\rangle + |10\rangle}{\sqrt{10}}, \qquad \Pr(B = 0) = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}.$$

Dividing this vector by $\sqrt{1/5}$ gives $(|00\rangle + |10\rangle)/\sqrt{2}$.

For $B = 1$, keep

$$\frac{2\,|01\rangle - 2\,|11\rangle}{\sqrt{10}}, \qquad \Pr(B = 1) = \frac{4}{10} + \frac{4}{10} = \frac{4}{5}.$$

Dividing by $\sqrt{4/5}$ gives $(|01\rangle - |11\rangle)/\sqrt{2}$. Therefore

Outcome on B	Probability	Normalized joint state	State of A
0	1/5	$\dfrac{\|00\rangle + \|10\rangle}{\sqrt{2}} = \|+\rangle\,\|0\rangle$	$\|+\rangle$
1	4/5	$\dfrac{\|01\rangle - \|11\rangle}{\sqrt{2}} = \|-\rangle\,\|1\rangle$	$\|-\rangle$

(b) Conditioned on $B = 1$, qubit A is $|-\rangle$. Thus

$$\Pr(A = 0 \mid B = 1) = \Pr(A = 1 \mid B = 1) = \frac{1}{2}.$$

The probability of $B = 1$ followed by $A = 0$ is

$$\Pr(B = 1)\Pr(A = 0 \mid B = 1) = \frac{4}{5} \cdot \frac{1}{2} = \frac{2}{5}.$$

(c) For $|\phi\rangle$, the conditional state of A is $|-\rangle$, so $H\,|-\rangle = |1\rangle$. The outcome is 1 with certainty.

For the product state $|\psi\rangle = |+\rangle \otimes (|0\rangle + 2\,|1\rangle)/\sqrt{5}$ from Problem 1, conditioning on $B = 1$ leaves the joint state $|+\rangle\,|1\rangle$. Hence $H\,|+\rangle = |0\rangle$, and the outcome on A is 0 with certainty.

Problem 5: Creating entanglement with a circuit

The gates, in order, are H_1, $\mathrm{CNOT}_{1\to2}$, $\mathrm{CNOT}_{2\to3}$, X_3, and H_1.

(a) Applying each gate to the current state gives

$$|\psi_1\rangle = \frac{|000\rangle + |100\rangle}{\sqrt{2}},$$
$$|\psi_2\rangle = \frac{|000\rangle + |110\rangle}{\sqrt{2}},$$
$$|\psi_3\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}},$$
$$|\psi_4\rangle = \frac{|001\rangle + |110\rangle}{\sqrt{2}},$$
$$|\psi_5\rangle = \frac{|001\rangle + |101\rangle + |010\rangle - |110\rangle}{2}.$$

For the final gate, $|001\rangle$ becomes $(|001\rangle + |101\rangle)/\sqrt{2}$, while $|110\rangle$ becomes $(|010\rangle - |110\rangle)/\sqrt{2}$.

(b) The only outcomes with nonzero probability are

Outcome	001	010	101	110
Probability	1/4	1/4	1/4	1/4

In every outcome, qubits 2 and 3 give different bits. The probability that they give the same bit is 0.

(c) Group the terms according to the first bit:

$$|\psi_5\rangle = \frac{1}{2}|0\rangle(|01\rangle + |10\rangle) + \frac{1}{2}|1\rangle(|01\rangle - |10\rangle).$$

Each outcome has probability $1/4 + 1/4 = 1/2$. After normalization,

Outcome on qubit 1	Probability	State of qubits 2,3
0	1/2	$(\|01\rangle + \|10\rangle)/\sqrt{2}$
1	1/2	$(\|01\rangle - \|10\rangle)/\sqrt{2}$

Both states are entangled. A product factorization would require $ac = bd = 0$, $ad = 1/\sqrt{2}$, and $bc = \pm 1/\sqrt{2}$. The nonzero values of ad and bc force all four factors to be nonzero, contradicting $ac = 0$.